ma_week2

✍️内容

> ♨ **Important**
>
> 记号说明：以下的符号表示 $\frac{\partial^2 f}{\partial x\partial y}$ 表示为 $\frac{\partial}{\partial x}\frac{\partial f}{\partial y}$，这与本书恰恰相反，但是我习惯了一般的教材上的符号，写的时候没有注意到。

9. 作自变量的变换，取 ξ，η，ζ 为新的自变量：

(1) $\xi = x$，$\eta = x^2 + y^2$，变换方程 $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$；

(2) $\xi = x$，$\eta = y - x$，$\zeta = z - x$，变换方程 $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

9 (1)

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}\cdot(2x)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial z}{\partial \eta}\cdot(2y)$$

于是

$$0 = y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y\left[\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}(2x) + \frac{\partial z}{\partial \eta}(2x)\right] = \pm\sqrt{\eta - \xi^2}\left(\frac{\partial z}{\partial \xi} + 4\xi\frac{\partial z}{\partial \eta}\right)$$

9 (2)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \zeta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{y} = \frac{\partial u}{\partial \eta}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial z} = \frac{\partial u}{\partial \zeta}$$

于是

$$0 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial \xi}$$

10. 作自变量和因变量的变换，取 u，v 为新的自变量，$w=w(u,v)$为新的因变量：

（1）设 $u=x+y$，$v=\frac{y}{x}$，$w=\frac{z}{x}$，变换方程

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0;$$

（2）设 $u=\frac{x}{y}$，$v=x$，$w=xz-y$，变换方程

$$y\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial y}=\frac{2}{x}.$$

10 (1)

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial (xw)}{\partial x}=w+x\frac{\partial w}{\partial x}=w+x\frac{\partial w}{\partial u}\frac{\partial u}{\partial x}+x\frac{\partial w}{\partial v}\frac{\partial v}{\partial x}\\
&= w+x\frac{\partial w}{\partial u}-\frac{y}{x}\frac{\partial w}{\partial v}\\
\frac{\partial z}{\partial y} &= \frac{\partial (xw)}{\partial y}=x\frac{\partial w}{\partial y}=x\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}+x\frac{\partial w}{\partial v}\frac{\partial v}{\partial y}\\
&= x\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}\\
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(w+x\frac{\partial w}{\partial u}-\frac{y}{x}\frac{\partial w}{\partial v}\right)=\frac{\partial w}{\partial x}+\frac{\partial w}{\partial u}+x\frac{\partial}{\partial x}\frac{\partial w}{\partial u}+\frac{y}{x^2}\frac{\partial w}{\partial v}-\frac{y}{x}\frac{\partial}{\partial x}\frac{\partial w}{\partial v}\\
&= 2\frac{\partial w}{\partial u}+x\frac{\partial^2 w}{\partial u^2}-\frac{y}{x}\frac{\partial^2 w}{\partial v\partial u}-\frac{y}{x}\frac{\partial^2 w}{\partial u\partial v}+\frac{y^2}{x^3}\frac{\partial^2 w}{\partial v^2}\\
\frac{\partial^2 z}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(x\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}\right)=\frac{\partial w}{\partial u}+x\left(\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial v\partial u}\cdot\left(-\frac{y}{x^2}\right)\right)+\left(\frac{\partial^2 w}{\partial u\partial v}+\frac{\partial^2 w}{\partial v^2}\cdot\left(-\frac{y}{x^2}\right)\right)\\
&= \frac{\partial w}{\partial u}+x\frac{\partial^2 w}{\partial u^2}-\frac{y}{x}\frac{\partial^2 w}{\partial v\partial u}+\frac{\partial^2 w}{\partial v\partial u}-\frac{y}{x^2}\frac{\partial^2 w}{\partial v^2}\\
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial y}\right)=x\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial v\partial u}+\frac{\partial^2 w}{\partial u\partial v}+\frac{\partial^2 w}{\partial v^2}\frac{1}{x}
\end{aligned}$$

于是

$$0=\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=\left(\frac{y}{x}+1\right)\left(\frac{\partial^2 w}{\partial v\partial u}-\frac{\partial^2 w}{\partial u\partial v}\right)+\frac{(x+y)^2}{x^3}\frac{\partial^2 w}{\partial v^2}$$

即

$$0=(x+y)\left(\frac{\partial^2 w}{\partial v\partial u}-\frac{\partial^2 w}{\partial u\partial v}\right)+\frac{(x+y)^2}{x^2}\frac{\partial^2 w}{\partial v^2}$$

即

$$0=u\left(\frac{\partial^2 w}{\partial v\partial u}-\frac{\partial^2 w}{\partial u\partial v}\right)+(1+v)^2\frac{\partial^2 w}{\partial v^2}$$

10 (2)

$$\frac{\partial z}{\partial y}=\frac{\partial \frac{w+y}{x}}{\partial y}=\frac{1}{x}\left(\frac{\partial w}{\partial y}+1\right)=\frac{1}{x}\left(\frac{\partial w}{\partial v}\overset{-\frac{x}{y^2}}{\cancel{\frac{\partial v}{\partial y}}}+\frac{\partial w}{\partial v}\overset{0}{\cancel{\frac{\partial v}{\partial y}}}+1\right)$$

$$=-\frac{1}{y^2}\frac{\partial w}{\partial u}+\frac{1}{x}$$

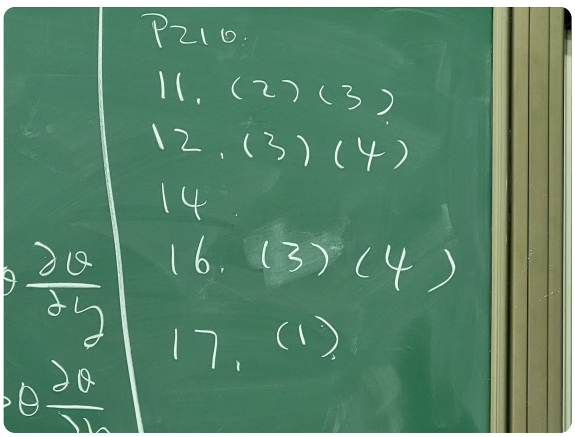
$$\frac{\partial^2 z}{\partial y^2}=\frac{2}{y^3}\frac{\partial w}{\partial u}+\frac{x^2}{y^4}\frac{\partial^2 w}{\partial u^2}$$

于是

$$0=y\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial y}-\frac{2}{x}=\frac{x^2}{y^3}\frac{\partial^2 w}{\partial u^2}$$

即

$$u^2\frac{\partial^2 w}{\partial u^2}=0$$



11. 求下列方程所确定的函数 $z=f(x,y)$ 的一阶和二阶偏导数：

(1) $e^{-xy}-2z+e^z=0$;

(2) $x+y+z=e^{-(x+y+z)}$;

(3) $xyz=x+y+z$;

11 (2)

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=-1,\qquad \frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial y^2}=0$$

11 (3)

$$\frac{\partial z}{\partial x}=-\frac{yz-1}{xy-1},\frac{\partial z}{\partial y}=-\frac{zx-1}{yz-1},\qquad \frac{\partial^2 z}{\partial x^2}=\frac{2(yz-1)y}{(xy-1)^2},\frac{\partial^2 z}{\partial y^2}=\frac{2(xz-1)x}{(xy-1)^2}$$

12. 求由下列方程所确定的函数的全微分 dz：

(3) $f(x+y+z, x^2+y^2+z^2)=0$;

(4) $f(x,y)+g(y,z)=0$.

12 (3)

$$\begin{aligned} 0 = df &= f_1 d(x+y+z) + f_2 d(x^2+y^2+z^2) \\ &= f_1(dx+dy+dz) + f_2(2xdx+2ydy+2zdz) \\ \implies dz &= \frac{-(f_1+2xf_2)dx-(f_1+2yf_2)dy}{f_1+2zf_2} \end{aligned}$$

12 (4)

$$\begin{aligned} 0 &= d(f(x,y)+g(y,z)) = f_1dx+f_2dy+g_1dy+g_2dz \\ \implies dz &= \frac{-f_1dx-(f_2+g_1)dy}{g_2} \end{aligned}$$

14. 设 $z=x^2+y^2$，其中 $y=f(x)$为由方程 $x^2-xy+y^2=1$ 所确定的隐函数，求$\frac{\mathrm{d}z}{\mathrm{d}x}$和$\frac{\mathrm{d}^2z}{\mathrm{d}x^2}$.

14

由隐函数方程可知：

$$2xdx - ydx - xdy + 2ydy = 0$$

再对 $z=x^2+y^2$ 求微分可以得到：

$$dz = 2xdx + 2ydy = \frac{2x^2-2y^2}{x-2y}dx \implies \frac{dz}{dx} = \frac{2x^2-2y^2}{x-2y}$$

在两边作用 $\frac{d}{dx}$ 得到：

$$\begin{aligned} \frac{d^2z}{dx^2} &= \left[\frac{4x}{x-2y} - \frac{2x^2-2y^2}{(x-2y)^2}\right] + \left[-\frac{4y}{x-2y} + \frac{4x^2-4y^2}{(x-2y)^2}\right]\frac{dy}{dx} \\ &= \frac{2(x^2-4xy+y^2)}{(x-2y)^2} + \frac{4(x^2-xy+y^2)}{(x-2y)^2}\frac{dy}{dx} \\ &= \frac{1}{(x-2y)^2}\left[2(1-3xy)+4\frac{dy}{dx}\right] \\ &= \frac{1}{(x-2y)^2}\left[2(1-3xy)+\frac{8x-4y}{x-2y}\right] \end{aligned}$$

16. 求下列方程组所确定的函数的导数或偏导数：

(3) $\begin{cases} u^2 - v = 3x + y, \\ u - 2v^2 = x - 2y, \end{cases}$ 求$\dfrac{\partial u}{\partial x}$，$\dfrac{\partial u}{\partial y}$，$\dfrac{\partial v}{\partial x}$，$\dfrac{\partial v}{\partial y}$；

(4) $\begin{cases} u = xyz, \\ x^2 + y^2 + z^2 = 1, \end{cases}$ 求$\dfrac{\partial^2 u}{\partial x^2}$，$\dfrac{\partial^2 u}{\partial y^2}$，$\dfrac{\partial^2 u}{\partial x \partial y}$.

16 (3)

$$\frac{\partial u}{\partial x} = \frac{12v-1}{8uv-1}, \qquad \frac{\partial u}{\partial y} = \frac{4v+3}{8uv-1}$$
$$\frac{\partial v}{\partial x} = \frac{3-2u}{8uv-1}, \qquad \frac{\partial v}{\partial y} = \frac{1+6u}{8uv-1}$$

16 (4)

计算得到

$$\begin{aligned}
d^2u =& \left(yz - \frac{x^2y}{z}\right)d^2x + \left(xz - \frac{xy^2}{z}\right)d^2y \\
&+ \left(ydz + zdy - \frac{2xy}{z}dx - \frac{x^2}{z}dy + \frac{x^2y}{z^2}dz\right)dx \\
&+ \left(xdz + zdx - \frac{y^2}{z}dx - \frac{2xy}{z}dy + \frac{xy^2}{z^2}dz\right)dy \\
=& \left(yz - \frac{x^2y}{z}\right)d^2x + \left(xz - \frac{xy^2}{z}\right)d^2y \\
&+ \left(y\frac{-xdx - ydy}{z} + zdy - \frac{2xy}{z}dx - \frac{x^2}{z}dy + \frac{x^2y}{z^2}\frac{-xdx - ydy}{z}\right)dx \\
&+ \left(x\frac{-xdx - ydy}{z} + zdx - \frac{y^2}{z}dx - \frac{2xy}{z}dy + \frac{xy^2}{z^2}\frac{-xdx - ydy}{z}\right)dy \\
=& \left(yz - \frac{x^2y}{z}\right)d^2x + \left(xz - \frac{xy^2}{z}\right)d^2y \\
&+ \frac{-xy(3z^2 + x^2)}{z^3}(dx)^2 + \frac{-xy(3z^2 + y^2)}{z^3}(dy)^2 \\
&+ \frac{-x^2y^2 - z^2(x^2 + y^2 - z^2)}{z^3}dydx + \frac{-x^2y^2 - z^2(x^2 + y^2 - z^2)}{z^3}dxdy
\end{aligned}$$

于是

$$\begin{aligned}
\frac{\partial^2 u}{\partial x \partial y} &= \frac{-x^2y^2 - z^2(x^2 + y^2 - z^2)}{z^3} \\
\frac{\partial^2 u}{\partial x^2} &= -\frac{xy(3z^2 + x^2)}{z^3} \\
\frac{\partial^2 u}{\partial y^2} &= -\frac{xy(3z^2 + y^2)}{z^3}
\end{aligned}$$

17. 下列方程组定义 z 为 x，y 的函数，求$\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$.

(1) $\begin{cases} x = \cos\theta\cos\varphi, \\ y = \cos\theta\sin\varphi, \\ z = \sin\theta; \end{cases}$　　(2) $\begin{cases} x = u + v, \\ y = u^2 + v^2, \\ z = u^3 + v^3. \end{cases}$

17 (1)

$$\frac{\partial z}{\partial x} = -\cot\theta\cos\varphi, \qquad \frac{\partial z}{\partial y} = -\cot\theta\sin\varphi$$

17 (2)

$$\frac{\partial z}{\partial x} = -2u^2 - 2v^2 - 9uv, \qquad \frac{\partial z}{\partial y} = \frac{5}{2}(u+v)$$