# ma_week1

## ✍️内容

数学分析简明教程（下册） 第二版 (邓东皋 尹小玲) (Z-Library) (1).pdf

1. 求下列函数的偏导数：

(1) $u = x^2\ln(x^2+y^2)$;

(2) $u = (x+y)\cos(xy)$;

(3) $u = \arctan\frac{y}{x}$;

(4) $u = xy + \frac{x}{y}$;

(5) $u = xye^{\sin(xy)}$;

(6) $u = x^y + y^x$.

2. 设

$$f(x,y)=\begin{cases} y\sin\frac{1}{x^2+y^2}, & x^2+y^2\neq 0, \\ 0, & x^2+y^2=0. \end{cases}$$

考察函数在(0,0)点的偏导数.

1 (1)

$$\frac{\partial u}{\partial x} = 2x\ln(x^2+y^2) + \frac{2x^3}{x^2+y^2}$$
$$\frac{\partial u}{\partial y} = \frac{2x^2y}{x^2+y^2}$$

1 (3)

$$\frac{\partial u}{\partial x} = \frac{1}{1+y^2/x^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$
$$\frac{\partial u}{\partial y} = \frac{1}{x}\cdot\frac{1}{1+y^2/x^2} = \frac{x}{x^2+y^2}$$

1 (6)

$$\frac{\partial u}{\partial x} = yx^{y-1} + y^x\ln y$$
$$\frac{\partial u}{\partial y} = x^y\ln x + xy^{x-1}$$

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$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=0$$
$$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0}\frac{y\sin(1/y^2)-0}{y}\text{不存在}$$

4. 求下列函数的全微分：

(1) $u=\sqrt{x^2+y^2+z^2}$；

(2) $u=xe^{yz}+e^{-x}+y$.

5. 求下列函数在给定点的全微分：

(1) $u=\dfrac{x}{\sqrt{x^2+y^2}}$在点(1,0)和(0,1)；

(2) $u=\ln(x+y^2)$在点(0,1)和(1,1)；

(3) $u=\sqrt[z]{\dfrac{x}{y}}$在点(1,1,1)；

(4) $u=x+(y-1)\arcsin\sqrt{\dfrac{x}{y}}$在点(0,1).

7. 证明函数

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$$

在(0,0)点连续且偏导数存在，但在此点不可微.

8. 证明函数

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$$

的偏导数存在，但偏导数在(0,0)点不连续，且在(0,0)点的任何邻域中无界，而$f$在原点(0,0)可微.

10. 设

$$f(x,y)=\begin{cases}\dfrac{1-e^{x(x^2+y^2)}}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$$

证明$f(x,y)$在(0,0)点可微，并求$df(0,0)$.

12. 设$|x|$，$|y|$很小，利用全微分推出下列各式的近似公式：

(1) $(1+x)^m(1+y)^n$；

(2) $\arctan\dfrac{x+y}{1+xy}$.

14. 设$\dfrac{\partial f}{\partial x}$在$(x_0,y_0)$存在，$\dfrac{\partial f}{\partial y}$在$(x_0,y_0)$连续，求证$f(x,y)$在$(x_0,y_0)$可微.

4 (2)

$$du=(e^{yz}-e^{-x})dx+(xze^{yz}+1)dy+(xye^{yz})dz$$

5 (2)

$$du=\frac{1}{x+y^2}dx+\frac{2y}{x+y^2}dy$$
$$du|_{(x,y)=(0,1)}=dx+2dy$$
$$du|_{(x,y)=(1,1)}=\frac{1}{2}dx+dy$$

5 (4)

$$du=u_xdx+u_ydy$$

在 $(0,1)$ 处，

$$u_x(0,1)=\lim_{\Delta x\to0}\frac{u(\Delta x,1)-u(0,1)}{\Delta x}=\lim_{\Delta x\to0}\frac{\Delta x}{\Delta x}=1$$
$$u_y(0,1)=\lim_{\Delta y\to0}\frac{u(0,1+\Delta y)-u(0,1)}{\Delta y}=\lim_{\Delta y\to0}\frac{0-0}{\Delta y}=0$$

于是

$$du=dx$$

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$$|f(x,y)|=\left|\frac{x^2y}{x^2+y^2}\right|\le\left|\frac{x^2y}{2|xy|}\right|\le\left|\frac{x}{2}\right|\to0(as\ (x,y)\to(0,0))$$

故 $f$ 在 $(0,0)$ 连续

$$\lim_{\Delta x\to0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to0}\frac{0-0}{\Delta x}=0\text{存在}$$
$$\lim_{\Delta y\to0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to0}\frac{0-0}{\Delta y}=0\text{存在}$$

故 $f$ 在 $(0,0)$ 处偏导数存在

$$f(\Delta x,\Delta y)-f(0,0)-0\cdot\Delta x-0\cdot\Delta y=\frac{(\Delta x)^2\cdot\Delta y}{(\Delta x)^2+(\Delta y)^2}$$

记 $\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$，令 $\Delta x=k\Delta y$ 则

$$\lim_{\rho\to 0}\frac{(\Delta x)^2\cdot\Delta y}{(\Delta x)^2+(\Delta y)^2}\cdot\frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\lim_{\Delta y\to 0}\frac{k^2(\Delta y)^3}{(1+k^2)^{3/2}\cdot(\Delta y)^3}=\frac{k^2}{(1+k^2)^{3/2}}\text{可取}k\text{使得非零}$$

于是

$$f(\Delta x,\Delta y)-f(0,0)-0\cdot\Delta x-0\cdot\Delta y=\frac{(\Delta x)^2\cdot\Delta y}{(\Delta x)^2+(\Delta y)^2}\neq o(\rho)$$

故 $f$ 在 $(0,0)$ 处不可微

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在 $(x,y)\neq(0,0)$ 处，$f$ 显然偏导数存在

$$f_x(x,y)=-\frac{2x\cos\left[\frac{1}{x^2+y^2}\right]}{x^2+y^2}+2x\sin\left[\frac{1}{x^2+y^2}\right]$$

$$f_y(x,y)=-\frac{2y\cos\left[\frac{1}{x^2+y^2}\right]}{x^2+y^2}+2y\sin\left[\frac{1}{x^2+y^2}\right]$$

在 $(x,y)=(0,0)$ 处

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{(\Delta x)^2\sin\frac{1}{(\Delta x)^2}}{\Delta x}=0$$

但

$$f_x(x,0)=-\frac{2\cos(1/x^2)}{x}+2x\sin(1/x^2)\to\infty(as\ x\to 0)$$

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{(\Delta y)^2\sin\frac{1}{(\Delta y)^2}}{\Delta y}=0$$

但

$$f_y(y,0)=-\frac{2\cos(1/y^2)}{y}+2y\sin(1/y^2)\to\infty(as\ y\to 0)$$

故 $f$ 偏导数在 $\mathbb{R}\times\mathbb{R}$ 上存在，但是在 $(0,0)$ 处不连续，且在 $(0,0)$ 的任何邻域中无界。

下面考察 $f$ 在 $(0,0)$ 处的可微性

$$f(\Delta x,\Delta y)-f(0,0)+f_x(0,0)\Delta x+f_y(0,0)\Delta y=((\Delta x)^2+(\Delta y)^2)\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}$$

记 $\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$，则

$$\lim_{\rho\to 0}\frac{\rho^2\sin\frac{1}{\rho^2}}{\rho}=0$$

故 $f$ 在 $(0,0)$ 处可微

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$(x,y)\to(0,0)$ 时

$$\lim_{(x,y)\to(0,0)}\frac{1-e^{x(x^2+y^2)}}{x^2+y^2}=\lim_{(x,y)\to(0,0)}\frac{x(x^2+y^2)}{x^2+y^2}=0$$

于是 $f$ 在 $(0,0)$ 处连续

$$f_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{1-e^{x^3}}{x^3}=-1$$

$$f_y(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0$$

记 $\rho=\sqrt{x^2+y^2}$，则

$$\lim_{\rho\to 0}\frac{f(x,y)-f_x(0,0)x-f_y(0,0)y-f(0,0)}{\sqrt{x^2+y^2}}=\lim_{\rho\to 0}\frac{1+x(x^2+y^2)-e^{x(x^2+y^2)}}{(x^2+y^2)^{3/2}}=\lim_{\rho\to 0}\frac{o(\rho^2)}{\rho^{3/2}}=0$$

故 $f$ 在 (0,0) 处可微

$$df=f_xdx+f_ydy$$

$$df(0,0)=f_x(0,0)dx+f_y(0,0)dy=-dx$$

12（2）

记 $f(x,y)=\arctan\frac{x+y}{1+xy}$，显然 $f$ 可微，则

$$f_x=\frac{1-y^2}{1+4xy+y^2+x^2+x^2y^2}$$
$$f_y=\frac{1-x^2}{1+4xy+y^2+x^2+x^2y^2}$$

于是在 $|x|,|y|$ 很小的时候

$$\begin{aligned}f(x,y)&=f(0,0)+f_x(0,0)x+f_y(0,0)y+o(\sqrt{x^2+y^2})\\&=x+y+o(\sqrt{x^2+y^2})\end{aligned}$$

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Proof:

令 $(x,y)\to(x_0,y_0)$，则

$$\begin{aligned}&f(x,y)-f(x_0,y_0)-\frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0)-\frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0)\\=&\left[f(x,y)-f(x,y_0)-\frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0)\right]+\left[f(x,y_0)-f(x_0,y_0)-\frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0)\right]\\=&\left[\frac{\partial f}{\partial y}(x,y_0)\cdot(y-y_0)-\frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0)\right]+o(|x-x_0|)+o(|y-y_0|)\\=&o(1)+o(|x-x_0|)+o(|y-y_0|)\\=&o(\sqrt{(x-x_0)^2+(y-y_0)^2})\end{aligned}$$

于是 $f$ 在 $(x_0,y_0)$ 处可微。

15. 求下列函数的所有二阶偏导数：

(1) $u=\ln\sqrt{x^2+y^2}$；

(2) $u=xy+\dfrac{y}{x}$；

(3) $u=x\sin(x+y)+y\cos(x+y)$；

(4) $u=e^{xy}$.

16. 求下列函数指定阶的偏导数：

(1) $u=x^3\sin y+y^3\sin x$，求$\dfrac{\partial^6 u}{\partial x^3\partial y^3}$；

(2) $u=\arctan\dfrac{x+y}{1-xy}$，求所有三阶偏导数；

(3) $u=\sin(x^2+y^2)$，求$\dfrac{\partial^3 u}{\partial x^3}$，$\dfrac{\partial^3 u}{\partial y^3}$；

19. 设 $f_x$，$f_y$ 在点$(x_0,y_0)$的某邻域内存在且在点$(x_0,y_0)$可微，则有

$$f_{xy}(x_0,y_0)=f_{yx}(x_0,y_0).$$

15 (1)

$$\begin{aligned}
\frac{\partial u}{\partial x}&=\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{x}{\sqrt{x^2+y^2}}=\frac{x}{x^2+y^2}\\
\frac{\partial u}{\partial y}&=\frac{y}{x^2+y^2}\\
\frac{\partial^2 u}{\partial x^2}&=\frac{1}{x^2+y^2}-\frac{2x^2}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}\\
\frac{\partial^2 u}{\partial y^2}&=\frac{x^2-y^2}{(x^2+y^2)^2}\\
\frac{\partial^2 u}{\partial x\partial y}&=\frac{\partial^2 u}{\partial y\partial x}=\frac{-2xy}{(x^2+y^2)^2}
\end{aligned}$$

15 (4)

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= ye^{xy} \\
\frac{\partial u}{\partial y} &= xe^{xy} \\
\frac{\partial^2 u}{\partial x^2} &= y^2 e^{xy} \\
\frac{\partial^2 u}{\partial y^2} &= x^2 e^{xy} \\
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial x e^{xy}}{\partial x} = e^{xy} + xye^{xy} = (1+xy)e^{xy} \\
\frac{\partial^2 u}{\partial y \partial x} &= (1+xy)e^{xy}
\end{aligned}
$$

16 (3)

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= 2x\cos(x^2+y^2) \\
\frac{\partial^2 u}{\partial x^2} &= 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2) \\
\frac{\partial^3 u}{\partial x^3} &= -4x\sin(x^2+y^2) - 8x\sin(x^2+y^2) - 8x^3\cos(x^2+y^2) \\
\frac{\partial u}{\partial y} &= 2y\cos(x^2+y^2) \\
\frac{\partial^2 u}{\partial y^2} &= 2\cos(x^2+y^2) - 4y^2\sin(x^2+y^2) \\
\frac{\partial^3 u}{\partial y^3} &= -4y\sin(x^2+y^2) - 8y\sin(x^2+y^2) - 8y^3\cos(x^2+y^2)
\end{aligned}
$$

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Proof:

不妨设 $(x_0, y_0) = (0,0)$，考虑 $(0,0)$ 的邻域 $U$ 中 $f_x, f_y$ 存在且在 $(0,0)$ 处可微，故对于 $(x,y) \in U$，$(x,y) \to (0,0)$ 时

$$
\begin{aligned}
f_{xy}(0,0) &= \lim_{y\to 0} \frac{f_x(0,y) - f_x(0,0)}{y} \\
&= \lim_{x\to 0}\lim_{y\to 0} \frac{f(x,y) - f(0,y) - f(x,0) - f(0,0)}{xy} \\
&= \lim_{x\to 0}\lim_{y\to 0} \frac{[f(x,y) - f(x,0)] - [f(0,y) - f(0,0)]}{xy} \\
&= \lim_{x\to 0}\lim_{y\to 0} \frac{f_y(x, \xi_1(x)\cdot y) - f_y(0, \xi_2 \cdot y)}{x} \qquad \text{微分中值定理，其中 } \overbrace{\xi_1(x)}^{\text{关于x的函数}},\ \overbrace{\xi_2}^{\text{常数}} \in [0,1] \\
&= \lim_{x\to 0}\lim_{y\to 0} \frac{[f_y(0,0) + f_{yx}(0,0)x + f_{yy}(0,0)\cdot\xi_1(x)y + o(\sqrt{x^2+\xi_1^2(x)y^2})] - [f_y(0,0) + f_{yx}(0,0)\cdot 0 + f_{yy}(0,0)}{x} \\
&= \lim_{x\to 0}\lim_{y\to 0} \frac{f_{yx}(0,0)x + f_{yy}(0,0)\cdot[\xi_1(x) - \xi_2]\cdot y + o(\sqrt{x^2+\xi_1^2(x)y^2}) + o(|y|)}{x} \\
&= \lim_{x\to 0} \frac{f_{yx}(0,0)\cdot x + o(|x|)}{x} \\
&= f_{yx}(0,0)
\end{aligned}
$$

1. 求下列函数的所有二阶偏导数：

（1） $u=f(ax,by)$；

（2） $u=f(x+y,x-y)$；

（5） $u=f(x^2+y^2+z^2)$；

（6） $u=f\left(x+y,xy,\dfrac{x}{y}\right)$.

2. 设 $z=\dfrac{y}{f(x^2-y^2)}$，其中 $f$ 是可微函数，验证

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}.$$

5. 验证下列各式：

（1） $u=\varphi(x^2+y^2)$，则 $y\dfrac{\partial u}{\partial x}-x\dfrac{\partial u}{\partial y}=0$；

（2） $u=y\varphi(x^2-y^2)$，则 $y\dfrac{\partial u}{\partial x}+x\dfrac{\partial u}{\partial y}=\dfrac{xu}{y}$；

1 (5)

$$\frac{\partial u}{\partial x}=2xf'(x^2+y^2+z^2)$$
$$\frac{\partial u}{\partial y}=2yf(x^2+y^2+z^2)$$
$$\frac{\partial u}{\partial z}=2zf(x^2+y^2+z^2)$$
$$\frac{\partial^2 u}{\partial x^2}=4x^2f''(x^2+y^2+z^2)+2f'(x^2+y^2+z^2)$$
$$\frac{\partial^2 u}{\partial y^2}=4y^2f''(x^2+y^2+z^2)+2f'(x^2+y^2+z^2)$$
$$\frac{\partial^2 u}{\partial z^2}=4z^2f''(x^2+y^2+z^2)+2f'(x^2+y^2+z^2)$$
$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}=4xyf''(x^2+y^2+z^2)$$
$$\frac{\partial^2 u}{\partial z\partial y}=\frac{\partial^2 u}{\partial y\partial z}=4yzf''(x^2+y^2+z^2)$$
$$\frac{\partial^2 u}{\partial x\partial z}=\frac{\partial^2 u}{\partial z\partial x}=4xzf''(x^2+y^2+z^2)$$

1 (6)

$$\begin{aligned}
\frac{\partial u}{\partial x} &= f_1\left(x+y, xy, \frac{x}{y}\right) + yf_2\left(x+y, xy, \frac{x}{y}\right) + \frac{1}{y}f_3\left(x+y, xy, \frac{x}{y}\right) \\
\frac{\partial u}{\partial y} &= f_1\left(x+y, xy, \frac{x}{y}\right) + xf_2\left(x+y, xy, \frac{x}{y}\right) - \frac{x}{y^2}f_3\left(x+y, xy, \frac{x}{y}\right) \\
\frac{\partial^2 u}{\partial x^2} &= f_{11} + yf_{12} + \frac{1}{y}f_{13} + yf_{21} + y^2 f_{22} + f_{23} + \frac{1}{y}f_{31} + f_{32} + \frac{1}{y^2}f_{33} \\
\frac{\partial^2 u}{\partial x \partial y} &= f_{11} + xf_{12} - \frac{x}{y^2}f_{13} + yf_{21} + xyf_{22} - \frac{x}{y}f_{23} + \frac{1}{y}f_{31} + \frac{x}{y}f_{32} - \frac{x}{y^3}f_{33} \\
\frac{\partial^2 u}{\partial y \partial x} &= f_{11} + yf_{12} + \frac{1}{y}f_{13} + f_2 + xf_{21} + xyf_{22} + \frac{x}{y}f_{23} - \frac{1}{y^2}f_3 - \frac{x}{y^2}f_{31} - \frac{x}{y}f_{32} - \frac{x}{y^3}f_{33} \\
\frac{\partial^2 u}{\partial y^2} &= f_{11} + xf_{12} - \frac{x}{y^2}f_{13} + xf_{21} + x^2 f_{22} - \frac{x^2}{y^2}f_{23} + \frac{2x}{y^3}f_3 - \frac{x}{y^2}f_{31} - \frac{x^2}{y^2}f_{32} + \frac{x^2}{y^4}f_{33}
\end{aligned}$$

2

$$\frac{\partial z}{\partial x} = \frac{\partial \frac{y}{f(x^2-y^2)}}{\partial x} = y\frac{\partial \frac{1}{f(x^2-y^2)}}{\partial f(x^2-y^2)}\frac{\partial f(x^2-y^2)}{\partial (x^2-y^2)}\frac{\partial (x^2-y^2)}{\partial x} = -\frac{2xyf'(x^2-y^2)}{f^2(x^2-y^2)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial \frac{y}{f(x^2-y^2)}}{\partial y} = \frac{1}{f(x^2-y^2)} + y\frac{\partial \frac{1}{f(x^2-y^2)}}{\partial f(x^2-y^2)}\frac{\partial f(x^2-y^2)}{\partial (x^2-y^2)}\frac{\partial (x^2-y^2)}{\partial y} = \frac{1}{f(x^2-y^2)} + \frac{2y^2f'(x^2-y^2)}{f^2(x^2-y^2)}$$

于是

$$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = -\frac{2yf'(x^2-y^2)}{f^2(x^2-y^2)} + \frac{1}{yf(x^2-y^2)} + \frac{2yf'(x^2-y^2)}{f^2(x^2-y^2)} = \frac{1}{yf(x^2-y^2)} = \frac{z}{y^2}$$

5（2）

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial y\varphi(x^2-y^2)}{\partial x} = 2xy\varphi'(x^2-y^2) \\
\frac{\partial u}{\partial y} &= \frac{\partial y\varphi(x^2-y^2)}{\partial y} = \varphi(x^2-y^2) - 2y^2\varphi'(x^2-y^2)
\end{aligned}$$

于是

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 2xy^2\varphi'(x^2-y^2) + x\varphi(x^2-y^2) - 2x^2\varphi'(x^2-y^2) = x\varphi(x^2-y^2) = \frac{xu}{y}$$