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1.(2) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sqrt[3]{n}}{\sqrt{n}+10}$, $\frac{\sqrt[3]{n}}{\sqrt{n}+10}=\frac{1}{\sqrt[6]{n}+\frac{10}{\sqrt[3]{n}}}$ 在 $n\geq 100$ 时递减趋于0，故 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sqrt[3]{n}}{\sqrt{n}+10}$ 收敛

1.(4) $\sum_{n=1}^{\infty}\frac{\ln^3(n+2)}{n+1}\cos\frac{n\pi}{2}=\sum_{n=1}^{\infty}\left[\frac{\ln^3(4n-1)}{4n-2}\cos\frac{(4n-3)\pi}{2}+\frac{\ln^3(4n)}{4n-1}\cos\frac{(4n-2)\pi}{2}+\frac{\ln^3(4n+1)}{4n}\cos\frac{(4n-1)\pi}{2}+\frac{\ln^3(4n+2)}{4n+1}\cos\frac{4n\pi}{2}\right]$

$=\sum_{n=1}^{\infty}\left[-\frac{\ln^3(4n)}{4n-1}+\frac{\ln^3(4n+2)}{4n+1}\right]=\sum_{n=1}^{\infty}(-1)^n\frac{\ln^3(2n)}{2n-1}$, $f(x)=\frac{\ln^3 x}{x-1}$, $x\geq 2$, $f'(x)=-\frac{\ln^2 x(-3x+x\ln x+3)}{(x-1)^2x}$, $x\geq e^3$ 时, $f'(x)\leq 0$, f 递减趋于0

故 $n\geq 2e^3$ 时，$\frac{\ln^3(2n)}{2n-1}$ 递减趋于0，故 $\sum_{n=1}^{\infty}(-1)^n\frac{\ln^3(2n)}{2n-1}$ 收敛

1.(6) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}+(-1)^{n-1}}=\sum_{n=1}^{\infty}\frac{1}{1+(-1)^{n-1}\sqrt{n}}=\frac{1}{(-1)^{n-1}\sqrt{n}}\sum_{n=1}^{\infty}\frac{1}{1+\frac{(-1)^{n-1}}{\sqrt{n}}}$

$=\frac{1}{(-1)^{n-1}\sqrt{n}}\sum_{n=1}^{\infty}\frac{1}{1-\frac{(-1)^n}{\sqrt{n}}}$

$\forall\varepsilon>0$, $\exists N>0$, $s.t.\ 1+\frac{(-1)^n}{\sqrt{n}}+\frac{1}{n}(1-\varepsilon)\leq\frac{1}{1-\frac{(-1)^n}{\sqrt{n}}}\leq 1+\frac{(-1)^n}{\sqrt{n}}+\frac{1}{n}(1+\varepsilon)$, $\forall n\geq N$

$\overline{\lim_{m\to\infty}}\frac{1}{(-1)^{n-1}\sqrt{n}}\sum_{n=N}^{m}\frac{1}{1-\frac{(-1)^n}{\sqrt{n}}}\leq\overline{\lim_{m\to\infty}}\frac{1}{(-1)^{n-1}\sqrt{n}}\sum_{n=N}^{m}1+\frac{(-1)^n}{\sqrt{n}}+\frac{1}{n}(1+\varepsilon)$

$=\overline{\lim_{m\to\infty}}\sum_{n=N}^{m}\frac{1}{(-1)^{n-1}\sqrt{n}}-\frac{1}{n}+\frac{1}{n}\frac{1}{(-1)^{n-1}\sqrt{n}}(1+\varepsilon)\to-\infty$ 故发散

1.(8) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\alpha}\sqrt[n]{n}}\ (\alpha>0)$，$\frac{1}{n^{\alpha}\sqrt[n]{n}}=\exp\left\{-\alpha\ln n-\frac{\ln n}{n}\right\}$, 在 $n>3$ 时, $\exp\left\{-\alpha\ln n-\frac{\ln n}{n}\right\}$ 递减趋于0, 故 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\alpha}\sqrt[n]{n}}$ 收敛

1.(10) $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{(2n-1)!!}{(2n)!!}\right)^p\ (p>0)$, 其中 $\frac{(2n+1)!!}{(2n+2)!!}=\frac{(2n-1)!!}{(2n)!!}\frac{2n+1}{2n+2}<\frac{(2n-1)!!}{(2n)!!}$, 故 $\frac{(2n-1)!!}{(2n)!!}$ 递减

$\lim_{n\to\infty}\frac{(2n-1)!!}{(2n)!!}=\lim_{n\to\infty}\frac{(2n)!}{(2n)!!\,(2n)!!}=\lim_{n\to\infty}\frac{(2n)!}{2^{2n}(n!)^2}=\lim_{n\to\infty}\frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{2^{2n}\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2}=0$, 故 $\left(\frac{(2n-1)!!}{(2n)!!}\right)^p$ 递减趋于0,

故 $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{(2n-1)!!}{(2n)!!}\right)^p$ 收敛

2.(1) $\sum_{n=1}^{\infty}\frac{(-1)^{\left[\frac{n}{4}\right]}}{n^{\alpha}}\ (\alpha>0)$，显然 $\left|\sum_{n=1}^{m}(-1)^{\left[\frac{n}{4}\right]}\right|\leq 4$ 有界 $\forall m\in\mathbb{N}$, $\frac{1}{n^{\alpha}}$ 递减趋于0，故 $\sum_{n=1}^{\infty}\frac{(-1)^{\left[\frac{n}{4}\right]}}{n^{\alpha}}$ 收敛

2.(3) $\sum_{n=1}^{\infty}\tan\left(\frac{(-1)^{n-1}\pi}{2\sqrt{n}}\right)\sin 2n=\lim_{m\to\infty}\sum_{n=1}^{m}\tan\left(\frac{(-1)^{2n-2}\pi}{2\sqrt{2n-1}}\right)\sin(4n-2)+\sum_{n=1}^{m}\tan\left(\frac{(-1)^{2n-1}\pi}{2\sqrt{2n}}\right)\sin(4n)$

$=\lim_{m\to\infty}\sum_{n=1}^{m}\tan\left(\frac{\pi}{2\sqrt{2n-1}}\right)\sin(4n-2)-\sum_{n=1}^{m}\tan\left(\frac{\pi}{2\sqrt{2n}}\right)\sin(4n)$

$\left|\sum_{n=1}^{m}\sin(4n-2)\right|=\left|\frac{\sin 1\sum_{n=1}^{m}\sin(4n-2)}{\sin 1}\right|=\left|\frac{\sum_{n=1}^{m}\frac{\cos(4n-3)-\cos(4n-1)}{2}}{\sin 1}\right|=\left|\frac{\cos 1-\cos(4m-1)}{2\sin 1}\right|\leq\frac{1}{\sin 1}$ 有界, $\forall m\in\mathbb{N}$

$\left|\sum_{n=1}^{m}\sin(4n)\right|=\left|\frac{\sin 1\sum_{n=1}^{m}\sin(4n)}{\sin 1}\right|=\left|\frac{\sum_{n=1}^{m}\frac{\cos(4n-1)-\cos(4n+1)}{2}}{\sin 1}\right|=\left|\frac{\cos 3-\cos(4m+1)}{2\sin 1}\right|\leq\frac{1}{\sin 1}$ 有界, $\forall m\in\mathbb{N}$

$\tan\left(\frac{\pi}{2\sqrt{2n-1}}\right)$, $\tan\left(\frac{\pi}{2\sqrt{2n-1}}\right)$ 递减趋于0，故 $\sum_{n=1}^{\infty}\tan\left(\frac{\pi}{2\sqrt{2n-1}}\right)\sin(4n-2)$，$\sum_{n=1}^{\infty}\tan\left(\frac{\pi}{2\sqrt{2n-1}}\right)\sin(4n)$ 收敛

故 $\sum_{n=1}^{\infty}\tan\left(\frac{(-1)^{n-1}\pi}{2\sqrt{n}}\right)\sin 2n=\lim_{m\to\infty}\sum_{n=1}^{m}\tan\left(\frac{\pi}{2\sqrt{2n-1}}\right)\sin(4n-2)-\sum_{n=1}^{m}\tan\left(\frac{\pi}{2\sqrt{2n}}\right)\sin(4n)$ 收敛

2.(5) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos^2 n}{n^{\alpha}}\ (\alpha>0)$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos^2 n}{n^{\alpha}}=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos 2n-1}{n^{\alpha}}$ 其中 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{-1}{n^{\alpha}}$ 收敛

考察 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos 2n}{n^{\alpha}}$

$$\left|\sum_{n=1}^{2m}(-1)^{n-1}\cos 2n\right|=\left|\sum_{n=1}^{m}(-1)^{2n-1}\cos 4n+\sum_{n=1}^{m}(-1)^{2n-2}\cos(4n-2)\right|=\left|-\sum_{n=1}^{m}\cos 4n+\sum_{n=1}^{m}\cos(4n-2)\right|$$

$$=\left|\frac{-\cos 1\sum_{n=1}^{m}\cos 4n+\cos 1\sum_{n=1}^{m}\cos(4n-2)}{\cos 1}\right|=\left|\frac{-\sum_{n=1}^{m}[\cos(4n-1)+\cos(4n+1)]+\sum_{n=1}^{m}[\cos(4n-3)+\cos(4n-1)]}{2\cos 1}\right|$$

$=\left|\frac{\cos 1-\cos(4m+1)}{2\cos 1}\right|\le\frac{1}{\cos 1}$ 有界，$\frac{1}{n^{\alpha}}$ 递减趋于0，故 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos 2n}{n^{\alpha}}$ 收敛，故 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos^2 n}{n^{\alpha}}$ 收敛

2.(7) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt[n]{n}}\sin\frac{1}{n^{\alpha}}\ (\alpha>0)$

$f(x)=\frac{1}{x^{\frac{1}{x}}}\sin\frac{1}{x^{\alpha}}, f'(x)=-\frac{1}{x^{\frac{1}{x}}}\cos\frac{1}{x^{\alpha}}\frac{\alpha+(1-\ln x)x^{\alpha}\tan\frac{1}{x^{\alpha}}}{x^{\alpha+1}}$，显然在 x 充分大的时候有 $f'(x)<0$ 恒成立

故 n 很大的时候 $\frac{1}{\sqrt[n]{n}}\sin\frac{1}{n^{\alpha}}$ 递减趋于0，故 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt[n]{n}}\sin\frac{1}{n^{\alpha}}$ 收敛

2.(10) $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{\ln n}\left(1+\frac{1}{n}\right)^{n}$，$\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n}}{\ln n}=\lim_{n\to\infty}\frac{e-\frac{e}{2n}+O\left(\frac{1}{n^2}\right)}{\ln n}=0$

$a_n:=\frac{\left(1+\frac{1}{n}\right)^{n}}{\ln n}=\exp\left\{n\ln\left(1+\frac{1}{n}\right)-\ln\ln n\right\}$，对于 $f(x)=x\ln\left(1+\frac{1}{x}\right)-\ln\ln x, f'(x)<0, \forall x>1$，

故 $\frac{\left(1+\frac{1}{n}\right)^{n}}{\ln n}$ 单调递减趋于0，故 $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{\ln n}\left(1+\frac{1}{n}\right)^{n}$ 收敛

2.(11) $\sum_{n=1}^{\infty}\frac{\sin n\sin n^2}{n^{\alpha}}\ (\alpha>0)$，$\left|\sum_{n=1}^{m}\sin n\sin n^2\right|=\left|\sum_{n=1}^{m}\frac{\cos(n^2-n)-\cos(n^2+n)}{2}\right|=\left|\frac{1-\cos(m^2+m)}{2}\right|\le 1$

$\frac{1}{n^{\alpha}}$ 递减趋于0，于是 $\sum_{n=1}^{\infty}\frac{\sin n\sin n^2}{n^{\alpha}}$ 收敛

3.(2) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\alpha}+\frac{1}{n}}$，$n>\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha+1}}$时，$\frac{1}{n^{\alpha}+\frac{1}{n}}$递减趋于0，故$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\alpha}+\frac{1}{n}}$收敛

3.(4) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\left[\sqrt{n}+(-1)^{n-1}\right]^{p}}\ (p>0)$

$n\to\infty$时，$\frac{(-1)^{n-1}}{\left[\sqrt{n}+(-1)^{n-1}\right]^{p}}=\frac{(-1)^{n-1}}{n^{\frac{p}{2}}}\left[1+\frac{(-1)^{n-1}}{\sqrt{n}}\right]^{-p}=\frac{(-1)^{n-1}}{n^{\frac{p}{2}}}\left[1+p\frac{(-1)^{n}}{\sqrt{n}}+O\left(\frac{1}{n}\right)\right]$

$=\frac{(-1)^{n-1}}{n^{\frac{p}{2}}}-\frac{p}{n^{\frac{p+1}{2}}}+O\left(\frac{1}{n^{1+\frac{p}{2}}}\right)$，

故由$\lim_{n\to\infty}\left|O\left(\frac{1}{n^{1+\frac{p}{2}}}\right)\Big/\frac{1}{n^{1+\frac{p}{2}}}\right|=A$有限，可知$\exists N>0, s.t.\ \forall n>N$有$\left|O\left(\frac{1}{n^{1+\frac{p}{2}}}\right)\right|\leq\frac{2A}{n^{1+\frac{p}{2}}}$

$$\left|\frac{(-1)^{n-1}}{\left[\sqrt{n}+(-1)^{n-1}\right]^{p}}-\frac{(-1)^{n-1}}{n^{\frac{p}{2}}}+\frac{p}{n^{\frac{p+1}{2}}}\right|\leq\frac{2A}{n^{1+\frac{p}{2}}},\forall n>N$$

$$\sum_{n=N}^{m}\left(\frac{(-1)^{n-1}}{\left[\sqrt{n}+(-1)^{n-1}\right]^{p}}-\frac{(-1)^{n-1}}{n^{\frac{p}{2}}}+\frac{p}{n^{\frac{p+1}{2}}}\right)\leq\sum_{n=N}^{m}\left|\frac{(-1)^{n-1}}{\left[\sqrt{n}+(-1)^{n-1}\right]^{p}}-\frac{(-1)^{n-1}}{n^{\frac{p}{2}}}+\frac{p}{n^{\frac{p+1}{2}}}\right|\leq\sum_{n=N}^{m}\frac{2A}{n^{1+\frac{p}{2}}}$$

$$\leq\sum_{n=1}^{\infty}\frac{2A}{n^{1+\frac{p}{2}}}<\infty$$

故$\lim_{m\to\infty}\sum_{n=N}^{m}\frac{(-1)^{n-1}}{\left[\sqrt{n}+(-1)^{n-1}\right]^{p}}$与$\lim_{m\to\infty}\sum_{n=N}^{m}\left(\frac{(-1)^{n-1}}{n^{\frac{p}{2}}}-\frac{p}{n^{\frac{p+1}{2}}}\right)$同敛散性

由于$\lim_{m\to\infty}\sum_{n=N}^{m}\frac{(-1)^{n-1}}{n^{\frac{p}{2}}}$收敛，$\lim_{m\to\infty}\sum_{n=N}^{m}\frac{p}{n^{\frac{p+1}{2}}}\begin{cases}\text{发散，当}p\leq 1\\\text{收敛，当}p>1\end{cases}$

故$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\left[\sqrt{n}+(-1)^{n-1}\right]^{p}}\begin{cases}\text{发散，当}p\leq 1\\\text{收敛，当}p>1\end{cases}$

$3.(6)\ \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}\left(1+\frac{(-1)^{n}}{n^{p}}\right)^{n}\ (p>0)$

①$p>1$时

$p>\frac{3}{2}$时,$n\to\infty$时,

$$\left(1+\frac{(-1)^{n}}{n^{p}}\right)^{n}=\exp\left\{n\ln\left(1+\frac{(-1)^{n}}{n^{p}}\right)\right\}=\exp\left\{n\left[\frac{(-1)^{n}}{n^{p}}+O\left(\frac{1}{n^{2p}}\right)\right]\right\}=\exp\left\{\frac{(-1)^{n}}{n^{p-1}}+O\left(\frac{1}{n^{2p-1}}\right)\right\}$$

$$=1+\frac{(-1)^{n}}{n^{p-1}}+O\left(\frac{1}{n^{2p-2}}\right)$$

$\frac{(-1)^{n-1}}{\sqrt{n}}\left(1+\frac{(-1)^{n}}{n^{p}}\right)^{n}=\frac{(-1)^{n-1}}{\sqrt{n}}-\frac{1}{n^{p-\frac{1}{2}}}+O\left(\frac{1}{n^{2p-\frac{3}{2}}}\right)$,故$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}\left(1+\frac{(-1)^{n}}{n^{p}}\right)^{n}$收敛

$1<p\le\frac{3}{2}$时, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}\left(1+\frac{(-1)^{n}}{n^{p}}\right)^{n}$发散

②$p=1$时

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}\left(1+\frac{(-1)^{n}}{n}\right)^{n}$,记$u_n=\frac{(-1)^{n-1}}{\sqrt{n}}\left(1+\frac{(-1)^{n}}{n}\right)^{n}$

$n\to\infty$时,

$$u_{2n}=\frac{(-1)^{2n-1}}{\sqrt{2n}}\left(1+\frac{(-1)^{2n}}{2n}\right)^{2n}=-\frac{1}{\sqrt{2n}}e\cdot e^{-\frac{1}{4n}+o\left(\frac{1}{n}\right)}=-\frac{1}{\sqrt{2n}}e\cdot(1+o(1))$$

$$u_{2n+1}=\frac{(-1)^{2n+1-1}}{\sqrt{2n+1}}\left(1+\frac{(-1)^{2n+1}}{2n+1}\right)^{2n+1}=\frac{1}{\sqrt{2n+1}}\left(1-\frac{1}{2n+1}\right)^{2n+1}=\frac{1}{\sqrt{2n+1}}e^{-1}\cdot e^{-\frac{1}{4n+2}+o\left(\frac{1}{n}\right)}$$

$$\le\frac{1}{\sqrt{2n}}e^{-1}\cdot e^{-\frac{1}{4n+2}+o\left(\frac{1}{n}\right)}=\frac{1}{\sqrt{2n}}e^{-1}\cdot(1+o(1))$$

$$u_{2n}+u_{2n+1}\le-\frac{1}{\sqrt{2n}}e\cdot(1+o(1))+\frac{1}{\sqrt{2n}}e^{-1}\cdot(1+o(1))=\frac{e^{-1}-e}{\sqrt{2n}}(1+o(1))$$

$$\sum_{k=2n}^{4n}u_k=\sum_{k=n}^{2n}u_{2k}+u_{2k+1}\le\sum_{k=n}^{2n}\frac{e^{-1}-e}{\sqrt{2n}}(1+o(1))\to-\infty$$

由柯西收敛准则, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}\left(1+\frac{(-1)^{n}}{n}\right)^{n}$发散

③$p<1$时

$$u_{2n}=\frac{-1}{\sqrt{2n}}\left(1+\frac{1}{(2n)^{p}}\right)^{2n}=\frac{-1}{\sqrt{2n}}\exp\{(2n)^{1-p}+o(n^{1-p})\}\to-\infty(as\ n\to\infty)$$

由柯西收敛准则, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}\left(1+\frac{(-1)^{n}}{n}\right)^{n}$发散

$$综上\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}\left(1+\frac{(-1)^{n}}{n}\right)^{n}\begin{cases}收敛,\ p>\frac{3}{2}时\\ 发散,\ 0<p\le\frac{3}{2}\end{cases}$$

$6.(1)\sum_{n=2}^{\infty}\frac{\cos nx}{n^p}\ (p>0)$

①$x=2k\pi,k\in\mathbb{Z}$时，$\sum_{n=2}^{\infty}\frac{\cos nx}{n^p}=\sum_{n=2}^{\infty}\frac{1}{n^p}\begin{cases}\text{绝对收敛，当}p>1\\\text{发散，当}p\leq 1\end{cases}$

②$x\neq 2k\pi,k\in\mathbb{Z}$时，$\left|\sum_{n=2}^{m}\cos nx\right|=\left|\frac{\sum_{n=2}^{m}\sin\frac{x}{2}\cos nx}{\sin\frac{x}{2}}\right|=\left|\frac{\sum_{n=2}^{m}\sin\left(n+\frac{1}{2}\right)x-\sin\left(n-\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\right|$

$=\left|\frac{\sin\left(m+\frac{1}{2}\right)x-\sin\frac{3}{2}x}{2\sin\frac{x}{2}}\right|\leq\left|\frac{1}{\sin\frac{x}{2}}\right|<\infty$，又$\frac{1}{n^p}$递减趋于0，故$\sum_{n=2}^{\infty}\frac{\cos nx}{n^p}\ (p>0)$收敛

下面我们判断条件收敛和绝对收敛

显然有$p>1$时$\sum_{n=2}^{\infty}\frac{\cos nx}{n^p}$绝对收敛

因为$\sum_{n=2}^{\infty}\left|\frac{\cos nx}{n^p}\right|\leq\sum_{n=2}^{\infty}\left|\frac{1}{n^p}\right|<\infty$

$p\leq 1$时，$\sum_{n=2}^{\infty}\frac{\cos nx}{n^p}$条件收敛

$$\sum_{n=2}^{\infty}\left|\frac{\cos nx}{n^p}\right|=\lim_{m\to\infty}\sum_{n=2}^{m}\frac{|\cos nx|}{n^p}\geq\lim_{m\to\infty}\sum_{n=2}^{m}\frac{|\cos nx|^2}{n^p}=\lim_{m\to\infty}\sum_{n=2}^{m}\frac{\frac{\cos 2nx+1}{2}}{n^p}=\lim_{m\to\infty}\frac{1}{2}\sum_{n=2}^{m}\frac{\cos 2nx+1}{n^p}$$

若$x=k\pi,k\in\mathbb{Z}$，则$\lim_{m\to\infty}\frac{1}{2}\sum_{n=2}^{m}\frac{\cos 2nx+1}{n^p}=\lim_{m\to\infty}\frac{1}{2}\sum_{n=2}^{m}\frac{2}{n^p}\to\infty$

若$x\neq k\pi,k\in\mathbb{Z}$，则$\lim_{m\to\infty}\frac{1}{2}\sum_{n=2}^{m}\frac{\cos 2nx+1}{n^p}=\lim_{m\to\infty}\frac{1}{2}\left(\sum_{n=2}^{m}\frac{\cos 2nx}{n^p}+\sum_{n=2}^{m}\frac{1}{n^p}\right)$前者收敛后者发散，故整体发散

综上：$\sum_{n=2}^{\infty}\frac{\cos nx}{n^p}\begin{cases}\text{绝对收敛，当}p>1\\\text{条件收敛，当}0<p\leq 1\text{且}x\neq 2k\pi,k\in\mathbb{Z}\\\text{发散，当}0<p\leq 1\text{且}x=2k\pi,k\in\mathbb{Z}\end{cases}$

$6.(3)\sum_{n=2}^{\infty}\ln\left(1+\frac{x^n}{n^p}\right)(p>0)$

①$|x|>1$时,$\ln\left(1+\frac{x^{2n}}{(2n)^p}\right)\to+\infty(as\ n\to\infty)$,故$\sum_{n=2}^{\infty}\ln\left(1+\frac{x^n}{n^p}\right)$发散

②$x=1$时，$\ln\left(1+\frac{1}{n^p}\right)=\frac{1}{n^p}+o\left(\frac{1}{n^p}\right)$,$p>1$时收敛且绝对收敛，$p\leq1$时发散

③$x=-1$时，$\ln\left(1+\frac{(-1)^n}{n^p}\right)=\frac{(-1)^n}{n^p}-\frac{1}{2n^{2p}}+o\left(\frac{1}{n^{2p}}\right)$,$p>\frac{1}{2}$时收敛，$p\leq\frac{1}{2}$时发散

$$\left|\ln\left(1+\frac{(-1)^n}{n^p}\right)\right|=\left|\frac{(-1)^n}{n^p}+o\left(\frac{1}{n^p}\right)\right|=\frac{1}{n^p}+o\left(\frac{1}{n^p}\right)，\text{故}\sum_{n=2}^{\infty}\ln\left(1+\frac{x^n}{n^p}\right)\begin{cases}\text{绝对收敛，当}p>1\\\text{条件收敛，当}\frac{1}{2}<p\leq1\end{cases}$$

④$|x|<1$时，$\lim_{n\to\infty}\frac{\left|\ln\left(1+\frac{x^n}{n^p}\right)\right|}{\frac{x^n}{n^p}}=1$,由比较判别法和阿贝尔判别法知收敛且绝对收敛

$6.(4)\sum_{n=1}^{\infty}\frac{\sin nx}{n^p+\frac{1}{n}}\ (p>0)$

①$x=k\pi,k\in\mathbb{Z}$时,$\sum_{n=1}^{\infty}\frac{\sin nx}{n^p+\frac{1}{n}}=0$收敛且绝对收敛

②$x\neq k\pi,k\in\mathbb{Z}$时,$\left|\sum_{n=1}^{m}\sin nx\right|=\left|\frac{\sum_{n=1}^{m}\sin\frac{x}{2}\sin nx}{\sin\frac{x}{2}}\right|=\left|\frac{\sum_{n=1}^{m}\cos\left(n-\frac{1}{2}\right)x-\cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\right|$

$=\left|\frac{\cos\frac{x}{2}-\cos\left(m+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\right|\leq\left|\frac{1}{\sin\frac{x}{2}}\right|<\infty$, 又$\frac{1}{n^p+\frac{1}{n}}$在$n$很大时递减趋于0，故$\sum_{n=1}^{\infty}\frac{\sin nx}{n^p+\frac{1}{n}}\ (p>0)$收敛

下面我们判断条件收敛和绝对收敛

$p>1$时，显然$\sum_{n=1}^{\infty}\left|\frac{\sin nx}{n^p+\frac{1}{n}}\right|\leq\sum_{n=1}^{\infty}\left|\frac{1}{n^p+\frac{1}{n}}\right|<\infty$

$p\leq1$时，$\sum_{n=1}^{\infty}\left|\frac{\sin nx}{n^p+\frac{1}{n}}\right|\geq\sum_{n=1}^{\infty}\frac{\sin^2 nx}{n^p+\frac{1}{n}}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1-\cos2nx}{n^p+\frac{1}{n}}=\frac{1}{2}\lim_{m\to\infty}\sum_{n=1}^{m}\frac{1}{n^p+\frac{1}{n}}-\frac{\cos2nx}{n^p+\frac{1}{n}}$前者发散后者收敛

故整体发散

综上：$\sum_{n=1}^{\infty}\frac{\sin nx}{n^p+\frac{1}{n}}\begin{cases}\text{绝对收敛，①}x=k\pi,k\in\mathbb{Z}\text{时，或②}x\neq k\pi,k\in\mathbb{Z}，p>1\text{时}\\\text{条件收敛，}x\neq k\pi,k\in\mathbb{Z}，0<p\leq1\text{时}\end{cases}$

7.正项级数$\sum_{n=1}^{\infty}(w_n-u_n)\leq\sum_{n=1}^{\infty}(v_n-u_n)<\infty$,故$\sum_{n=1}^{\infty}(w_n-u_n)$收敛

由于$\sum_{n=1}^{\infty}u_n$收敛,故$\sum_{n=1}^{\infty}w_n=\sum_{n=1}^{\infty}(w_n-u_n)+\sum_{n=1}^{\infty}u_n$收敛

8.$\forall\varepsilon>0,\ \exists N_1>0,s.t.|u_n|<\varepsilon,\ \forall n>N_1$

$\exists N_2>0,s.t.\left|\sum_{k=2m-1}^{2n}u_k\right|=\left|\sum_{k=m}^{n}(u_{2k-1}+u_{2k})\right|<\varepsilon,\ \forall n>m>N_2$

故$\forall m>n>2\max\{N_1,N_2\}$,有

$$\left|\sum_{k=m}^{n}u_k\right|\leq\left|u_{2\left[\frac{m}{2}\right]-1}\right|+\left|\sum_{k=2\left[\frac{m}{2}\right]-1}^{2\left[\frac{n}{2}\right]}u_k\right|+\left|u_{2\left[\frac{n}{2}\right]+1}\right|\leq 3\varepsilon$$

由柯西收敛准则可知$\sum_{n=1}^{\infty}u_n$收敛

9.由于交错级数$\sum_{n=1}^{\infty}(-1)^{n-1}u_n\ (u_n>0,\forall n)$条件收敛但不绝对收敛

故$n\to\infty$时,$S_{2n}-S_{2n-1}=O(1),S_{2n-1}+S_{2n}\to+\infty$

故$S_{2n-1}\to+\infty,S_{2n}\to+\infty$

故$\lim\limits_{n\to\infty}\frac{S_{2n-1}}{S_{2n}}=\lim\limits_{n\to\infty}\frac{S_{2n}+O(1)}{S_{2n}}=1$

10.(1)反例：$u_n=\dfrac{(-1)^n}{\sqrt{n}},v_n=1+\dfrac{(-1)^n}{\sqrt{n}}\to 1$

但$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}\left(1+\frac{(-1)^n}{\sqrt{n}}\right)=\lim_{m\to\infty}\sum_{n=1}^{m}\frac{(-1)^n}{\sqrt{n}}+\lim_{m\to\infty}\sum_{n=1}^{m}\frac{1}{n}$

前者收敛后者发散，故发散

10.(2)正确，证明如下：

$$\forall\varepsilon>0,\exists N>0,s.t.\begin{cases}\displaystyle\sum_{k=m}^{n}|u_k|\le\frac{2}{3}\varepsilon,\forall n>m>N\\[2ex]\dfrac{1}{2}<v_n<\dfrac{3}{2},\forall n>N\end{cases}$$

故$\forall n>m>N,\displaystyle\sum_{k=m}^{n}|v_ku_k|\le\varepsilon$.由柯西收敛准则知$\displaystyle\sum_{n=1}^{\infty}v_nu_n$绝对收敛

10.(3)反例：$u_n=\dfrac{(-1)^{n-1}}{n}$,但$\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-1)^{n-1}}{n}=\sum_{n=1}^{\infty}\frac{1}{n}\to+\infty$

10.(4)正项级数$\displaystyle\sum_{n=1}^{\infty}u_n$收敛蕴含$\displaystyle\sum_{n=1}^{\infty}u_{2n-1},\sum_{n=1}^{\infty}u_{2n}$收敛

故$\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}u_n=\sum_{n=1}^{\infty}u_{2n-1}-\sum_{n=1}^{\infty}u_{2n}$收敛

10.(5)反例：$u_n=\begin{cases}0,\text{若}n\text{为奇数}\\[1ex]\dfrac{1}{n},\text{若}n\text{为偶数}\end{cases}$

则级数$\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}u_n=\sum_{n=1}^{\infty}(-1)^{2n-1}u_{2n}=-\sum_{n=1}^{\infty}\frac{1}{2n}\to-\infty$

10.(6)反例：$u_n=\dfrac{(-1)^n}{n},v_n=\dfrac{(-1)^n}{n}+\dfrac{1}{n\ln n}$,显然$\displaystyle\lim_{n\to\infty}\frac{v_n}{u_n}=\lim_{n\to\infty}\frac{\frac{(-1)^n}{n}+\frac{1}{n\ln n}}{\frac{(-1)^n}{n}}=\lim_{n\to\infty}\frac{1+\frac{(-1)^n}{\ln n}}{1}=1$

故$u_n\sim v_n$,但$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}+\frac{1}{n\ln n}=\lim_{m\to\infty}\sum_{n=1}^{m}\frac{(-1)^n}{n}+\sum_{n=1}^{m}\frac{1}{n\ln n}$前者收敛后者发散故整体发散

11.(1) 不能，比如$u_n=\frac{(-1)^n}{\sqrt{n}}$

11.(2) 不能，比如我们考虑这样构造$\{u_n\}$：

$$1,-\frac{1}{2},-\frac{1}{2},\frac{1}{\sqrt[3]{2}},-\frac{1}{2\sqrt[3]{2}},-\frac{1}{2\sqrt[3]{2}},\frac{1}{\sqrt[3]{3}},-\frac{1}{2\sqrt[3]{3}},-\frac{1}{2\sqrt[3]{3}},\cdots,\frac{1}{\sqrt[3]{n}},-\frac{1}{2\sqrt[3]{n}},-\frac{1}{2\sqrt[3]{n}},\cdots$$

注意到$\underline{\lim}_{m\to\infty}\sum_{n=1}^{m}u_n=0$，$\overline{\lim}_{m\to\infty}\sum_{n=1}^{m}u_n=0$故$\sum_{n=1}^{\infty}u_n=0$收敛

但是考虑$\{u_n^3\}$：

$$1,-\frac{1}{8},-\frac{1}{8},\frac{1}{2},-\frac{1}{16},-\frac{1}{16},\frac{1}{3},-\frac{1}{24},-\frac{1}{24},\cdots,\frac{1}{n},-\frac{1}{8n},-\frac{1}{8n},\cdots$$

$$\sum_{n=1}^{3m}u_n^3=\sum_{n=1}^{m}u_{3n-2}^3+u_{3n-1}^3+u_{3n}^3=\sum_{n=1}^{m}\frac{1}{n}-\frac{1}{8n}-\frac{1}{8n}=\frac{3}{4}\sum_{n=1}^{m}\frac{1}{n}$$

故$\lim_{m\to\infty}\sum_{n=1}^{3m}u_n^3=\frac{3}{4}\lim_{m\to\infty}\sum_{n=1}^{m}\frac{1}{n}\to+\infty$

11.(3) 由于$\sum_{n=1}^{\infty}u_n^2$收敛，故$\lim_{n\to\infty}u_n^2=0$，故$\lim_{n\to\infty}|u_n|=0$

故$\forall\varepsilon>0$，$\exists N>0$，$s.t.\begin{cases}\left|\sum_{k=m}^{n}u_k^2\right|<2\varepsilon,\forall n>m>N\\|u_n|<\frac{1}{2},\forall n>N\end{cases}$

故$\left|\sum_{k=m}^{n}|u_n^3|\right|\le\frac{1}{2}\left|\sum_{k=m}^{n}u_k^2\right|\le\varepsilon$，由柯西收敛准则知：级数$\sum_{n=1}^{\infty}u_n^3$绝对收敛

1.我们用比较原理来证：

正项级数$\sum_{n=1}^{\infty}(u_n+|u_n|)\le\sum_{n=1}^{\infty}2|u_n|=2\sum_{n=1}^{\infty}|u_n|<\infty$

级数$\sum_{n=1}^{\infty}(-|u_n|)$收敛，故$\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}(u_n+|u_n|)+\sum_{n=1}^{\infty}(-|u_n|)$收敛