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$$1.(2)\sum_{n=1}^{\infty}\frac{1}{9n^2-3n-2}=\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n+1)}=\frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{3n-2}-\frac{1}{3n+1}=\frac{1}{3}$$

$$(4)\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}=\frac{1}{4}$$

$$(5)\sum_{n=1}^{\infty}\left(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}\right)=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+2}+\sqrt{n+1}}-\frac{1}{\sqrt{n+1}+\sqrt{n}}=1-\sqrt{2}$$

$$(6)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n(n+2)}\left(\sqrt{n}+\sqrt{n+2}\right)}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(\sqrt{n+2}-\sqrt{n}\right)}{\sqrt{n(n+2)}}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}-\frac{(-1)^{n+1}}{\sqrt{n+2}}=\frac{2-\sqrt{2}}{4}$$

$$(8)\sum_{n=1}^{\infty}\arctan\frac{4}{4n^2+3}=\sum_{n=1}^{\infty}\arctan\frac{\left(n+\frac{1}{2}\right)-\left(n-\frac{1}{2}\right)}{1+\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}=\sum_{n=1}^{\infty}\arctan\left(n+\frac{1}{2}\right)-\arctan\left(n-\frac{1}{2}\right)=\frac{\pi}{2}-\arctan\frac{1}{2}=\arctan 2$$

$$(12)\sum_{n=2}^{\infty}(-1)^{n-1}\arccos\frac{n\sqrt{n^2-4}+1}{n^2-1}=\sum_{n=2}^{\infty}(-1)^{n-1}\arccos\left(\frac{1}{n-1}\frac{1}{n+1}+\sqrt{1-\left(\frac{1}{n-1}\right)^2}\sqrt{1-\left(\frac{1}{n+1}\right)^2}\right)$$

$$=\sum_{n=2}^{\infty}(-1)^{n-1}\left(\arccos\frac{1}{n-1}-\arccos\frac{1}{n+1}\right)=-\arccos 1+\arccos\frac{1}{2}=\frac{\pi}{3}$$

2.(1)①$x\geq 1$时, $\sum_{n=1}^{m}nx^n\geq\sum_{n=1}^{m}n=\frac{m(m-1)}{2}\to\infty(as\ m\to\infty)$故$\sum_{n=1}^{\infty}nx^n$发散

②$0\leqslant|x|<1$时, $\sum_{n=1}^{m}nx^n=x\left(\sum_{n=1}^{m}x^n\right)'=x\left(\frac{x-x^{m+1}}{1-x}\right)'=x\left[\frac{x-x^{m+1}}{(1-x)^2}+\frac{1-(m+1)x^m}{1-x}\right]$

$=x\frac{x-x^{m+1}+1-x-(m+1)x^m+(m+1)x^{m+1}}{(1-x)^2}=\frac{mx^{m+1}-(m+1)x^m+x}{(1-x)^2}\to\frac{x}{(1-x)^2}(as\ m\to\infty)$,故$\sum_{n=1}^{\infty}nx^n$收敛

③$x\leq-1$时,取$y=-x\geq 1$,对于$m\geq 1$, $\sum_{n=2m-1}^{2m}nx^n=\sum_{n=2m-1}^{2m}n(-1)^ny^n=(2m)y^{2m}-(2m-1)y^{2m-1}$

$\geq(2m)y^{2m}-(2m-1)y^{2m}=y^{2m}\geq 1$,由柯西收敛准则: $\sum_{n=1}^{\infty}nx^n$发散.

(3) ①$0\leqslant|x|<1$时，$\left(\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n}x^n\right)'=\sum_{n=1}^{m}(-1)^{n-1}x^{n-1}=\sum_{n=0}^{m}(-x)^n=\frac{1-(-x)^{m+1}}{1+x}$

$$\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n}x^n=\int_0^x\left(\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n}t^n\right)'dt+\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n}x^n\Bigg|_{x=0}=\int_0^x\frac{1-(-t)^{m+1}}{1+t}dt$$

$$=\int_0^x\frac{1}{1+t}dt+(-1)^m\int_0^x\frac{t^{m+1}}{1+t}dt=\ln(1+x)+(-1)^m\int_0^x\frac{t^{m+1}}{1+t}dt$$

$$\left|\overline{\lim_{m\to\infty}}(-1)^m\int_0^x\frac{t^{m+1}}{1+t}dt\right|=\left|\overline{\lim_{m\to\infty}}\int_0^x\frac{t^{m+1}}{1+t}dt\right|\leq\left|\overline{\lim_{m\to\infty}}x^{m+1}\int_0^x\frac{1}{1+t}dt\right|=\left|\overline{\lim_{m\to\infty}}x^{m+1}\ln(1+x)\right|\overset{0\leqslant|x|<1}{=}0$$

故 $\lim\limits_{m\to\infty}\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n}x^n=\ln(1+x)$，故 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n=\ln(1+x)$收敛

②$x=-1$时，$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n=-\sum_{n=1}^{\infty}\frac{1}{n}$发散

③$x=1$时，$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$由Dirichlet判别法收敛

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=\lim_{m\to\infty}\sum_{n=1}^{m}\frac{1}{2n-1}-\frac{1}{2n}=\lim_{m\to\infty}\sum_{n=1}^{2m}\frac{1}{n}-\sum_{n=1}^{m}\frac{1}{2n}-\sum_{n=1}^{m}\frac{1}{2n}=\lim_{m\to\infty}\sum_{n=1}^{2m}\frac{1}{n}-\sum_{n=1}^{m}\frac{1}{n}=\ln 2$$

④$|x|>1$时，对于$m\geq1$，$\sum_{n=2m-1}^{2m+1}\frac{(-1)^{n-1}x^n}{n}=\frac{x^{2m-1}}{2m-1}-\frac{x^{2m}}{2m}+\frac{x^{2m+1}}{2m+1}$

$\geq2\sqrt{\frac{1}{4m^2-1}}x^{2m}-\frac{x^{2m}}{2m}\geq\frac{x^{2m}}{2m}\to+\infty\,(as\ m\to\infty)$.故$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$发散.

(6) $x>0$时，$\sum_{n=1}^{m}ne^{-nx}=-\left(\sum_{n=1}^{m}e^{-nx}\right)'=-\left(\frac{e^{-x}-e^{-(m+1)x}}{1-e^{-x}}\right)'=-\left(\frac{1-e^{-mx}}{e^x-1}\right)'$

$=\frac{e^x-(m+1)e^{-(m-1)x}+me^{-mx}}{(e^x-1)^2}\to\frac{e^x}{(e^x-1)^2}\,(as\ m\to\infty)$，故$\sum_{n=1}^{\infty}ne^{-nx}=\frac{e^x}{(e^x-1)^2}$收敛

$x\leqslant0$时，$\sum_{n=1}^{\infty}ne^{-nx}\geq\sum_{n=1}^{\infty}n\to+\infty$发散

(10) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n(2n-1)}=\lim_{m\to\infty}\sum_{n=1}^{m}\frac{(-1)^{n-1}x^{2n}}{n(2n-1)}$ 是偶函数,故不妨只考虑$x\ge 0$的情况,$x<0$的时候对称.

①$0\le x\le 1$时, $\begin{cases}\left(\sum_{n=1}^{m}\frac{(-1)^{n-1}x^{2n}}{n(2n-1)}\right)'=2\sum_{n=1}^{m}\frac{(-1)^{n-1}x^{2n-1}}{2n-1},\\ \left(\sum_{n=1}^{m}\frac{(-1)^{n-1}x^{2n}}{n(2n-1)}\right)''=2\sum_{n=1}^{m}(-1)^{n-1}x^{2n-2}=2\sum_{n=0}^{m}(-x^2)^{n}=\frac{2-2(-x^2)^{m+1}}{1+x^2}\end{cases}$

故 $\left(\sum_{n=1}^{m}\frac{(-1)^{n-1}x^{2n}}{n(2n-1)}\right)'=\int_0^x\left(\sum_{n=1}^{m}\frac{(-1)^{n-1}t^{2n}}{n(2n-1)}\right)''dt+\left(\sum_{n=1}^{m}\frac{(-1)^{n-1}x^{2n}}{n(2n-1)}\right)'\Bigg|_{x=0}=\int_0^x\frac{2-2(-t^2)^{m+1}}{1+t^2}dt+2\sum_{n=1}^{m}\frac{(-1)^{n-1}x^{2n-1}}{2n-1}\Bigg|_{x=0}$

$=2\int_0^x\frac{1}{1+t^2}dt-2\int_0^x\frac{(-t^2)^{m+1}}{1+t^2}dt=2\arctan x-2\int_0^x\frac{(-t^2)^{m+1}}{1+t^2}dt$

故 $\sum_{n=1}^{m}\frac{(-1)^{n-1}x^{2n}}{n(2n-1)}=\int_0^x\left(\sum_{n=1}^{m}\frac{(-1)^{n-1}t^{2n}}{n(2n-1)}\right)'dt+\sum_{n=1}^{m}\frac{(-1)^{n-1}x^{2n}}{n(2n-1)}\Bigg|_{x=0}=\int_0^x\left(\sum_{n=1}^{m}\frac{(-1)^{n-1}t^{2n}}{n(2n-1)}\right)'dt$

$=\int_0^x\left[2\arctan t-2\int_0^t\frac{(-s^2)^{m+1}}{1+s^2}ds\right]dt=2\int_0^x\arctan t\,dt+2\int_0^x\int_0^t\frac{(-s^2)^{m+1}}{1+s^2}ds\,dt=2x\arctan x-\ln(1+x^2)+2\int_0^x\int_0^t\frac{(-s^2)^{m+1}}{1+s^2}ds\,dt$

而 $\left|\overline{\lim_{m\to\infty}}\int_0^x\int_0^t\frac{(-s^2)^{m+1}}{1+s^2}ds\,dt\right|\le\overline{\lim_{m\to\infty}}\int_0^x\int_0^t\left|\frac{(-s^2)^{m+1}}{1+s^2}\right|ds\,dt\le\overline{\lim_{m\to\infty}}\int_0^1\int_0^t\frac{s^{2m+2}}{1+s^2}ds\,dt$

$\overset{\delta\in(0,1)}{\le}\overline{\lim_{m\to\infty}}\int_0^\delta\int_0^t\frac{s^{2m+2}}{1+s^2}ds\,dt+\overline{\lim_{m\to\infty}}\int_\delta^1\int_0^t\frac{s^{2m+2}}{1+s^2}ds\,dt\le\overline{\lim_{m\to\infty}}\int_0^\delta\int_0^t s^{2m+2}ds\,dt+\overline{\lim_{m\to\infty}}\int_\delta^1\int_0^t 1\,ds\,dt$

$\le\overline{\lim_{m\to\infty}}\int_0^\delta t^{2m+3}dt+\overline{\lim_{m\to\infty}}\int_\delta^1 1\,dt\le\overline{\lim_{m\to\infty}}\delta^{2m+4}+(1-\delta)=1-\delta$, 由$\delta\in(0,1)$任意性,令$\delta\to 1^-$,

故 $\lim_{m\to\infty}\int_0^x\int_0^t\frac{(-s^2)^{m+1}}{1+s^2}ds\,dt=0$.于是 $\lim_{m\to\infty}\sum_{n=1}^{m}\frac{(-1)^{n-1}x^{2n}}{n(2n-1)}=2x\arctan x-\ln(1+x^2)$收敛.

②$x>1$时,对于$m\ge 1$, $\sum_{n=2m-1}^{2m+1}\frac{(-1)^{n-1}x^{2n}}{n(2n-1)}=\frac{x^{4m-2}}{(2m-1)(4m-3)}-\frac{x^{4m}}{2m(4m-1)}+\frac{x^{4m+2}}{(2m+1)(4m+1)}$

$\ge 2\sqrt{\frac{1}{(2m-1)(2m+1)}\cdot\frac{1}{(4m-3)(4m+1)}}x^{4m}-\frac{x^{4m}}{2m(4m-1)}=2\sqrt{\frac{1}{4m^2-1}\cdot\frac{1}{(4m-1)^2-4}}x^{4m}-\frac{x^{4m}}{2m(4m-1)}$

$\ge\frac{2x^{4m}}{2m(4m-1)}-\frac{x^{4m}}{2m(4m-1)}=\frac{x^{4m}}{2m(4m-1)}\to+\infty(as\ m\to+\infty)$

由柯西收敛准则: $\lim_{m\to\infty}\sum_{n=1}^{m}\frac{(-1)^{n-1}x^{2n}}{n(2n-1)}$ 发散.

3.(1) $\sum_{n=1}^{\infty}\frac{1}{2n-1}=\lim_{m\to\infty}\sum_{n=1}^{m}\frac{1}{2n-1}=\lim_{m\to\infty}\left(\sum_{n=1}^{2m}\frac{1}{n}-\sum_{n=1}^{m}\frac{1}{2n}\right)=\lim_{m\to\infty}\left(\sum_{n=1}^{2m}\frac{1}{n}-\frac{1}{2}\sum_{n=1}^{m}\frac{1}{n}\right)$

$=\lim_{m\to\infty}\left(\ln 2m+c+\varepsilon_{2m}-\frac{1}{2}(\ln m+c+\varepsilon_m)\right)=\lim_{m\to\infty}\frac{1}{2}\ln m+\ln 2+\frac{c}{2}\to+\infty$发散

(2) $\sum_{n=1}^{\infty}\frac{1}{2n^2+n-1}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)(n+1)}=2\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+2)}=\frac{2}{3}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n+2}\right)$

$=\frac{2}{3}\lim_{m\to\infty}\sum_{n=1}^{m}\left(\frac{1}{2n-1}-\frac{1}{2n+2}\right)=\frac{2}{3}\lim_{m\to\infty}\left[\left(\sum_{n=1}^{2m}\frac{1}{n}-\sum_{n=1}^{m}\frac{1}{2n}\right)-\sum_{n=2}^{m+1}\frac{1}{2n}\right]=\frac{2}{3}\lim_{m\to\infty}\left(\sum_{n=1}^{2m}\frac{1}{n}-\sum_{n=1}^{m}\frac{1}{2n}-\sum_{n=1}^{m+1}\frac{1}{2n}+\frac{1}{2}\right)$

$=\frac{2}{3}\lim_{m\to\infty}\left(\ln 2m+c+\varepsilon_{2m}-\frac{1}{2}(\ln m+c+\varepsilon_m)-\frac{1}{2}(\ln(m+1)+c+\varepsilon_{m+1})+\frac{1}{2}\right)=\frac{2}{3}\lim_{m\to\infty}\ln\frac{2m}{\sqrt{m(m+1)}}+\frac{1}{3}=\frac{2\ln 2+1}{3}$

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$1.(2)\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+1}}\le\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}<\infty$,故收敛

$1.(4)\sqrt[n]{n}=e^{\frac{\ln n}{n}}\le e^{\frac{1}{e}},\sum_{n=1}^{\infty}\frac{1}{n\sqrt[n]{n}}\ge\sum_{n=1}^{\infty}\frac{1}{ne^{\frac{1}{e}}}\to+\infty$,故发散

$1.(6)\sum_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^p-1\right](p>0)$,由伯努利不等式:$\sum_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^p-1\right]\ge\sum_{n=1}^{\infty}\frac{p}{n}\to+\infty$,故发散

$1.(8)\sum_{n=1}^{\infty}\frac{1}{[\ln(\sqrt[n]{n}+2)]^n}\le\sum_{n=1}^{\infty}\frac{1}{\ln^n 3}=\frac{1}{\ln 3-1}$

$2.(2)\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})^p\ln\frac{n+1}{n+2}$

$n\to\infty$时,$(\sqrt{n+1}-\sqrt{n})^p\ln\frac{n+1}{n+2}=(\sqrt{n+1}-\sqrt{n})^p\ln\left(1-\frac{1}{n+2}\right)$

$=\left(\frac{1}{\sqrt{n}}\right)^p\left(\frac{1}{1+\sqrt{1+\frac{1}{n}}}\right)^p\left(\frac{1}{n+2}+o\left(\frac{1}{n+2}\right)\right)=n^{-\frac{p}{2}}\left(\frac{1}{1+1+\frac{1}{2n}+o\left(\frac{1}{n}\right)}\right)^p\left(\frac{1}{n}+o\left(\frac{1}{n}\right)\right)$

$=2^{-p}n^{-\frac{p}{2}}\left(1+\frac{1}{4n}+o\left(\frac{1}{n}\right)\right)^{-p}\left(\frac{1}{n}+o\left(\frac{1}{n}\right)\right)=2^{-p}n^{-\frac{p}{2}}\left(1-\frac{p}{4n}+o\left(\frac{1}{n}\right)\right)\left(\frac{1}{n}+o\left(\frac{1}{n}\right)\right)=2^{-p}n^{-\frac{p}{2}-1}+o\left(n^{-\frac{p}{2}-1}\right)$

故 $\lim_{n\to\infty}\frac{(\sqrt{n+1}-\sqrt{n})^p\ln\frac{n+1}{n+2}}{n^{-\frac{p}{2}-1}}=2^{-p}>0$

由于$p>0,-\frac{p}{2}-1<-1$,故$\sum_{n=1}^{\infty}n^{-\frac{p}{2}-1}$收敛

故$\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})^p\ln\frac{n+1}{n+2}$收敛

$2.(4)\sum_{n=1}^{\infty}\frac{1-e^{-\frac{1}{n^p}}}{n^p}$

$n\to\infty$时,$\frac{1-e^{-\frac{1}{n^p}}}{n^p}=n^{-p}\left(1-e^{-\frac{1}{n^p}}\right)=n^{-p}\left(\frac{1}{n^p}+o\left(\frac{1}{n^p}\right)\right)=1+o(1)$

故 $\exists N>0,s.t.\forall n>N$,有$\frac{1-e^{-\frac{1}{n^p}}}{n^p}>\frac{1}{2}$,故$\sum_{n=1}^{\infty}\frac{1-e^{-\frac{1}{n^p}}}{n^p}$发散

$3.(4)\ \sum_{n=1}^{\infty}\frac{n!}{\left(n+\frac{1}{n}\right)^{n}}$

$\overline{\lim_{n\to\infty}}\sqrt[n]{\frac{n!}{\left(n+\frac{1}{n}\right)^{n}}}=\overline{\lim_{n\to\infty}}\frac{\sqrt[n]{n!}}{n+\frac{1}{n}}=\overline{\lim_{n\to\infty}}\frac{\frac{n}{e}+o(n)}{n+\frac{1}{n}}=\frac{1}{e}<1$，故 $\sum_{n=1}^{\infty}\frac{n!}{\left(n+\frac{1}{n}\right)^{n}}$ 收敛

$3.(5)\ \sum_{n=1}^{\infty}\frac{n!\,2^{n}}{n^{n}}$

$\overline{\lim_{n\to\infty}}\sqrt[n]{\frac{n!\,2^{n}}{n^{n}}}=\overline{\lim_{n\to\infty}}\frac{2\sqrt[n]{n!}}{n}=\overline{\lim_{n\to\infty}}\frac{\frac{2n}{e}+o(n)}{n}=\frac{2}{e}<1$，故 $\sum_{n=1}^{\infty}\frac{n!\,2^{n}}{n^{n}}$ 收敛

$3.(6)\ \sum_{n=1}^{\infty}\frac{n!\,3^{n}}{n^{n}}$

$\overline{\lim_{n\to\infty}}\sqrt[n]{\frac{n!\,3^{n}}{n^{n}}}=\overline{\lim_{n\to\infty}}\frac{3\sqrt[n]{n!}}{n}=\overline{\lim_{n\to\infty}}\frac{\frac{3n}{e}+o(n)}{n}=\frac{3}{e}>1$，故 $\sum_{n=1}^{\infty}\frac{n!\,3^{n}}{n^{n}}$ 发散

$3.(9)\ \sum_{n=1}^{\infty}\left(\frac{2n-1}{3n+1}\right)^{n}$

$\overline{\lim_{n\to\infty}}\sqrt[n]{\left(\frac{2n-1}{3n+1}\right)^{n}}=\overline{\lim_{n\to\infty}}\frac{2n-1}{3n+1}=\frac{2}{3}<1$，故 $\sum_{n=1}^{\infty}\left(\frac{2n-1}{3n+1}\right)^{n}$ 收敛

$3.(12)\ \sum_{n=1}^{\infty}\frac{(2n^{2}-n)^{\frac{n+1}{2}}}{(3n^{3}+2n)^{\frac{n}{3}}}$

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\frac{(2n^{2}-n)^{\frac{n+1}{2}}}{(3n^{3}+2n)^{\frac{n}{3}}}}=\overline{\lim_{n\to\infty}}\frac{(2n^{2}-n)^{\frac{1}{2}}}{(3n^{3}+2n)^{\frac{1}{3}}}=\overline{\lim_{n\to\infty}}\frac{\left(2-\frac{1}{n}\right)^{\frac{1}{2}}}{\left(3+2\frac{1}{n^{2}}\right)^{\frac{1}{3}}}=\overline{\lim_{n\to\infty}}\frac{2^{\frac{1}{2}}\left(1-\frac{1}{2n}\right)^{\frac{1}{2}}}{3^{\frac{1}{3}}\left(1+\frac{2}{3n^{2}}\right)^{\frac{1}{3}}}=\frac{2^{\frac{1}{2}}}{3^{\frac{1}{3}}}=0.980561<1$$

故 $\sum_{n=1}^{\infty}\frac{(2n^{2}-n)^{\frac{n+1}{2}}}{(3n^{3}+2n)^{\frac{n}{3}}}$ 收敛

$$4.(2)\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^{p}\ (p>0)$$

$$n\to\infty\text{时},\left(\frac{(2n-1)!!}{(2n)!!}\Big/\frac{(2n+1)!!}{(2n+2)!!}\right)^{p}=\left(\frac{2n+2}{2n+1}\right)^{p}=\left(1+\frac{1}{2n+1}\right)^{p}$$

$$=\left(1+\frac{1}{2n}\frac{1}{1+\frac{1}{2n}}\right)^{p}=\left(1+\frac{1}{2n}\left(1-\frac{1}{2n}+o\left(\frac{1}{n}\right)\right)\right)^{p}=\left(1+\frac{1}{2n}-\frac{1}{4n^{2}}+o\left(\frac{1}{n^{2}}\right)\right)^{p}$$

$$=1+\frac{p}{2n}-\frac{p}{4n^{2}}+o\left(\frac{1}{n^{2}}\right)+\frac{p(p-1)}{2}\left(\frac{1}{2n}-\frac{1}{4n^{2}}+o\left(\frac{1}{n^{2}}\right)\right)^{2}+o\left(\frac{1}{n^{2}}\right)$$

$$=1+\frac{p}{2n}-\frac{p}{4n^{2}}+\frac{p(p-1)}{8n^{2}}+o\left(\frac{1}{n^{2}}\right)=1+\frac{p}{2n}+\frac{p(p-3)}{8n^{2}}+o\left(\frac{1}{n^{2}}\right)$$

$$\left(\frac{(2n-1)!!}{(2n)!!}\Big/\frac{(2n+1)!!}{(2n+2)!!}\right)^{p}=\left(\frac{2n+2}{2n+1}\right)^{p}=\left(1+\frac{1}{2n+1}\right)^{p}=\left(1+\frac{1}{2n}+o\left(\frac{1}{n}\right)\right)^{p}$$

$$=1+\frac{p}{2n}+o\left(\frac{1}{n}\right)$$

当$p<2$时，$\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^{p}$发散，当$p>2$时，$\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^{p}$收敛

$$p=2\text{时},\overline{\lim_{n\to\infty}}\,n\ln n\left[\left(\frac{(2n-1)!!}{(2n)!!}\Big/\frac{(2n+1)!!}{(2n+2)!!}\right)^{2}-1-\frac{1}{n}\right]=\overline{\lim_{n\to\infty}}\,n\ln n\left[\left(1+\frac{1}{2n+1}\right)^{2}-1-\frac{1}{n}\right]$$

$$=\overline{\lim_{n\to\infty}}\frac{\left(1+\frac{1}{2n+1}\right)^{2}-1-\frac{1}{n}}{n\ln n}=\overline{\lim_{x\to\infty}}\frac{\left(1+\frac{1}{2x+1}\right)^{2}-1-\frac{1}{x}}{x\ln x}$$

$$=\overline{\lim_{x\to\infty}}\frac{p\left(1+\frac{1}{2x+1}\right)\left(-\frac{2}{(2x+1)^{2}}\right)+\frac{1}{x^{2}}}{1+\ln x}=\overline{\lim_{x\to\infty}}\frac{-p\left(\frac{2}{(2x+1)^{2}}+\frac{2}{(2x+1)^{3}}\right)+\frac{1}{x^{2}}}{1+\ln x}=0<1$$

由$Bertrand$判别法：$\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^{2}$发散

4.(4) $\sum_{n=1}^{\infty}\frac{n!e^n}{n^{n+p}}\ (p>0)$

$$\frac{n!e^n}{n^{n+p}}\Big/\frac{(n+1)!e^{n+1}}{(n+1)^{n+1+p}}=\frac{1}{e}\frac{(n+1)^{n+p}}{n^{n+p}}=\frac{1}{e}\left(1+\frac{1}{n}\right)^{n+p}=e^{(n+p)\ln\left(1+\frac{1}{n}\right)-1}=e^{(n+p)\left(\frac{1}{n}-\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right)-1}$$

$$=e^{\frac{p-\frac{1}{2}}{n}+o\left(\frac{1}{n}\right)}=1+\frac{p-\frac{1}{2}}{n}+o\left(\frac{1}{n}\right)$$

当 $p<\frac{3}{2}$ 时, $\sum_{n=1}^{\infty}\frac{n!e^n}{n^{n+p}}$ 发散, 当 $p>\frac{3}{2}$ 时, $\sum_{n=1}^{\infty}\frac{n!e^n}{n^{n+p}}$ 收敛

$p=\frac{3}{2}$ 时, $\overline{\lim_{n\to\infty}}\, n\ln n\left[\left(\frac{n!e^n}{n^{n+\frac{3}{2}}}\Big/\frac{(n+1)!e^{n+1}}{(n+1)^{n+\frac{5}{2}}}\right)-1-\frac{1}{n}\right]=\overline{\lim_{n\to\infty}}\, n\ln n\left[e^{\left(n+\frac{3}{2}\right)\ln\left(1+\frac{1}{n}\right)-1}-1-\frac{1}{n}\right]$

$$=\overline{\lim_{n\to\infty}}\frac{e^{\left(n+\frac{3}{2}\right)\ln\left(1+\frac{1}{n}\right)-1}-1-\frac{1}{n}}{n\ln n}=\overline{\lim_{x\to\infty}}\frac{e^{\left(x+\frac{3}{2}\right)\ln\left(1+\frac{1}{x}\right)-1}-1-\frac{1}{x}}{x\ln x}$$

$$=\overline{\lim_{x\to\infty}}\frac{\left[\ln\left(1+\frac{1}{x}\right)+\left(x+\frac{3}{2}\right)\left(-\frac{1}{x^2}\right)\frac{1}{1+\frac{1}{x}}\right]e^{\left(x+\frac{3}{2}\right)\ln\left(1+\frac{1}{x}\right)-1}+\frac{1}{x^2}}{1+\ln x}$$

$$=\overline{\lim_{x\to\infty}}\frac{\left[\ln\left(1+\frac{1}{x}\right)-\frac{x+\frac{3}{2}}{x^2+x}\right]e^{\left(x+\frac{3}{2}\right)\ln\left(1+\frac{1}{x}\right)-1}+\frac{1}{x^2}}{1+\ln x}=0<1$$

由 *Bertrand* 判别法: $\sum_{n=1}^{\infty}\frac{n!e^n}{n^{n+\frac{3}{2}}}$ 发散

5.(2) $\sum_{n=9}^{\infty}\frac{1}{n\ln n(\ln\ln n)^p}$ 为正项级数, 由于 $f(x)=\frac{1}{x\ln x(\ln\ln x)^p}$ 在 $[9,+\infty)$ 单调递减

故可以使用积分判别法

$$\int_9^{+\infty}\frac{dx}{x\ln x(\ln\ln x)^p}=\int_9^{+\infty}\frac{d\ln x}{\ln x(\ln\ln x)^p}=\int_9^{+\infty}\frac{d\ln\ln x}{(\ln\ln x)^p}=\int_{\ln\ln 9}^{+\infty}\frac{dt}{t^p}$$

当 $p>1$ 时, 收敛; $0<p\le 1$ 时, 发散

5.(3) $\sum_{n=9}^{\infty}\frac{1}{n(\ln n)^p(\ln\ln n)^q}$ 为正项级数, 由于 $f(x)=\frac{1}{x(\ln x)^p(\ln\ln x)^q}$ 在 $[9,+\infty)$ 单调递减

故可以使用积分判别法

$$I=\int_9^{+\infty}\frac{dx}{x(\ln x)^p(\ln\ln x)^q}=\int_9^{+\infty}\frac{d\ln x}{(\ln x)^p(\ln\ln x)^q}=\int_9^{+\infty}\frac{d\ln x}{(\ln x)^p(\ln\ln x)^q}$$

$p=1$ 时, 由 (2) 可知: 当 $q>1$ 时, 收敛; $0<q\le 1$ 时, 发散

$p\ne 1$ 时, $I=\int_{2\ln 3}^{+\infty}\frac{dt}{t^p\ln^q t}$, $\frac{1}{\ln^q x}$ 不影响阶

$p<1$ 时, $\int_{2\ln 3}^{+\infty}\frac{dt}{t^p\ln^q t}$ 发散; $p>1$ 时, $\int_{2\ln 3}^{+\infty}\frac{dt}{t^p\ln^q t}$ 收敛

$8.(1)\sum_{n=1}^{m}\sqrt{u_n v_n} \le \left(\sum_{n=1}^{m}u_n\sum_{n=1}^{m}v_n\right)^{\frac{1}{2}}$,令 $m\to\infty$,则 $\lim_{m\to\infty}\sum_{n=1}^{m}\sqrt{u_n v_n} \le \lim_{m\to\infty}\left(\sum_{n=1}^{m}u_n\sum_{n=1}^{m}v_n\right)^{\frac{1}{2}}$ 收敛

$8.(2)\forall p>1$,证明 $\sum_{n=1}^{\infty}\sqrt[p]{u_n^p+v_n^p}$ 收敛

$\sum_{n=1}^{m}\sqrt[p]{u_n^p+v_n^p} \le \sum_{n=1}^{m}\sqrt[p]{(u_n+v_n)^p} = \sum_{n=1}^{m}u_n + \sum_{n=1}^{m}v_n$ 收敛

$8.(3)\forall \mu+\nu\ge 1,\mu>0,\nu>0$,证明 $\sum_{n=1}^{\infty}u_n^{\mu}v_n^{\nu}$ 收敛

$\forall 0<\varepsilon<1,\ \exists N>0,s.t.\forall m_2>m_1>N$,有 $\sum_{n=m_1}^{m_2}u_n\le\varepsilon,\ \sum_{n=m_1}^{m_2}v_n\le\varepsilon$

则 $\sum_{n=m_1}^{m_2}u_n^{\mu}v_n^{\nu} \overset{0<u_n,v_n<1}{\le} \sum_{n=m_1}^{m_2}u_n^{\frac{\mu}{\mu+\nu}}v_n^{\frac{\nu}{\mu+\nu}} \le \left(\sum_{n=m_1}^{m_2}u_n\right)^{\frac{\mu}{\mu+\nu}}\left(\sum_{n=m_1}^{m_2}v_n\right)^{\frac{\nu}{\mu+\nu}} \le \varepsilon^{\frac{\mu}{\mu+\nu}}\varepsilon^{\frac{\nu}{\mu+\nu}}=\varepsilon$

由柯西收敛准则: $\sum_{n=m_1}^{m_2}u_n^{\mu}v_n^{\nu}$ 收敛.

10.①$l>1$ 时, $\lim_{n\to\infty}u_n=l>1$,则 $\exists N\in\mathbb{N},s.t.\forall n>N,u_n>\frac{1+l}{2}>1$

则 $\lim_{n\to\infty}\frac{\frac{1}{n^{u_n}}}{\frac{1}{n^{\frac{1+l}{2}}}}=0$,故 $\sum_{n=1}^{\infty}\frac{1}{n^{u_n}}$ 收敛.

②$l<1$ 时, $\lim_{n\to\infty}u_n=l<1$,则 $\exists N\in\mathbb{N},s.t.\forall n>N,u_n<\frac{1+l}{2}<1$

则 $\lim_{n\to\infty}\frac{\frac{1}{n^{u_n}}}{\frac{1}{n^{\frac{1+l}{2}}}}=+\infty$,故 $\sum_{n=1}^{\infty}\frac{1}{n^{u_n}}$ 发散.

③令 $u_n\equiv1,\forall n$,则 $\sum_{n=1}^{\infty}\frac{1}{n^{u_n}}$ 发散

我们知道 $\sum_{n=2}^{\infty}\frac{1}{n\ln^2 n}$ 收敛.取 $u_1=1$,对于 $n\ge2$,

取 $n^{u_n}=n\ln^2 n\Rightarrow u_n\ln n=\ln n+2\ln\ln n\Rightarrow u_n=\frac{\ln n+2\ln\ln n}{\ln n}\to1$

这样就有 $\sum_{n=1}^{\infty}\frac{1}{n^{u_n}}$ 收敛.故 $l=1$ 时不能判断.