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$$1.(2)\int_0^1 \ln x dx = \lim_{\varepsilon\to0^+}\int_\varepsilon^1 \ln x dx = \lim_{\varepsilon\to0^+}(-\varepsilon\ln\varepsilon-(1-\varepsilon)) = -1$$

$$(3)\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \lim_{\substack{\varepsilon\to0^+\\ \delta\to0^+}}\int_\varepsilon^{1-\delta}\frac{dx}{\sqrt{x(1-x)}} \overset{x=\sin^2 t}{=} \lim_{\substack{\varepsilon\to0^+\\ \delta\to0^+}}\int_\varepsilon^{\frac{\pi}{2}-\delta}\frac{2\sin t\cos t dt}{\sin t\cos t} = 2\lim_{\substack{\varepsilon\to0^+\\ \delta\to0^+}}\left(\frac{\pi}{2}-\delta-\varepsilon\right) = \pi$$

$$(4)\int_0^1 \frac{dx}{(2+x)\sqrt{1-x}} = \lim_{\varepsilon\to0^+}\int_0^{1-\varepsilon}\frac{dx}{(2+x)\sqrt{1-x}} = -2\lim_{\varepsilon\to0^+}\int_0^{1-\varepsilon}\frac{d\sqrt{1-x}}{3-\left(\sqrt{1-x}\right)^2}$$

$$=-\frac{1}{\sqrt3}\lim_{\varepsilon\to0^+}\ln\left|\frac{\sqrt3+\sqrt{1-x}}{\sqrt3-\sqrt{1-x}}\right|\Bigg|_0^{1-\varepsilon} = \frac{1}{\sqrt3}\ln\frac{\sqrt3+1}{\sqrt3-1}$$

$$(5)\int_0^1 \cot x dx = \int_0^1\frac{\cos x}{\sin x}dx = \lim_{\varepsilon\to0^+}\int_\varepsilon^1 d\ln\sin x = \lim_{\varepsilon\to0^+}(\ln\sin1-\ln\sin\varepsilon)\to-\infty \text{积分发散}$$

$$(8)\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}} = \lim_{\substack{\varepsilon\to0^+\\ \delta\to0^+}}\int_{a+\varepsilon}^{b-\delta}\frac{dx}{\sqrt{(x-a)(b-x)}} = \lim_{\substack{\varepsilon\to0^+\\ \delta\to0^+}}\int_{a+\varepsilon}^{b-\delta}\frac{d(x-a)}{\sqrt{(x-a)(b-a-(x-a))}}$$

$$=\lim_{\substack{\varepsilon\to0^+\\ \delta\to0^+}}\int_\varepsilon^{b-a-\delta}\frac{dx}{\sqrt{x(b-a-x)}} = \lim_{\substack{\varepsilon\to0^+\\ \delta\to0^+}}\int_\varepsilon^{b-a-\delta}\frac{dx}{\sqrt{\left(\frac{b-a}{2}\right)^2-\left(x-\frac{b-a}{2}\right)^2}} = \lim_{\substack{\varepsilon\to0^+\\ \delta\to0^+}}\arcsin\frac{2x-(b-a)}{b-a}\Bigg|_\varepsilon^{b-a-\delta}$$

$$=\pi$$

$$(10)\int_{-1}^1 \frac{x\arcsin x}{\sqrt{1-x^2}}dx = 2\int_0^1\frac{x\arcsin x}{\sqrt{1-x^2}}dx = 2\lim_{\varepsilon\to0^+}\int_0^{1-\varepsilon}\frac{x\arcsin x}{\sqrt{1-x^2}}dx \overset{x=\sin t}{=} 2\lim_{\varepsilon\to0^+}\int_0^{\frac{\pi}{2}-\varepsilon}\frac{t\sin t}{\cos t}\cos t dt$$

$$=2\lim_{\varepsilon\to0^+}\int_0^{\frac{\pi}{2}-\varepsilon} t\sin t dt = -2\lim_{\varepsilon\to0^+}\int_0^{\frac{\pi}{2}-\varepsilon} t d\cos t = -2\lim_{\varepsilon\to0^+}\left[t\cos t\Big|_0^{\frac{\pi}{2}-\varepsilon}-\int_0^{\frac{\pi}{2}-\varepsilon}\cos t dt\right]$$

$$=2\lim_{\varepsilon\to0^+}\int_0^{\frac{\pi}{2}-\varepsilon}\cos t dt = 2\lim_{\varepsilon\to0^+}\sin\left(\frac{\pi}{2}-\varepsilon\right) = 2$$

2.(1) $\lim\limits_{x\to0^+}x^{\frac{1}{2}}|\ln x|^p=\lim\limits_{x\to0^+}x^{\frac{1}{2}}\left|2\ln x^{\frac{1}{2}}\right|^p=2^p\lim\limits_{x\to0^+}x|\ln x|^p=2^p\lim\limits_{x\to+\infty}\frac{\ln^p x}{x}=2^p\lim\limits_{x\to+\infty}\frac{x^p}{e^x}\overset{洛}{=}2^p\lim\limits_{x\to+\infty}\frac{px^{p-1}}{e^x}=\cdots=0$

故$\int_0^1|\ln x|^p dx$收敛

(2) $\int_0^1\frac{dx}{|\ln x|^p}=\int_0^{+\infty}t^{-p}e^{-t}dt$,问题集中在$t=0$处

$\int_0^{+\infty}t^{-p}e^{-t}dt=\int_0^{\delta}t^{-p}e^{-t}dt+\int_{\delta}^{+\infty}t^{-p}e^{-t}dt\le\int_0^{\delta}t^{-p}e^{-t}dt+\delta^{-p}\int_{\delta}^{+\infty}e^{-t}dt=\int_0^{\delta}t^{-p}e^{-t}dt+\delta^{-p}e^{-\delta}$

$\int_0^{+\infty}t^{-p}e^{-t}dt=\int_0^{\delta}t^{-p}e^{-t}dt+\int_{\delta}^{+\infty}t^{-p}e^{-t}dt\ge\int_0^{\delta}t^{-p}e^{-t}dt$

$\int_0^{x}t^{-p}e^{-t}dt$随$x\uparrow$而$\uparrow$,故$\int_0^{+\infty}t^{-p}e^{-t}dt$与$\int_0^{\delta}t^{-p}e^{-t}dt$同敛散性

$\lim\limits_{t\to0^+}t^p(t^{-p}e^{-t})=1$,故$p<1$时,$\int_0^{+\infty}t^{-p}e^{-t}dt$收敛;$p\ge1$时,$\int_0^{+\infty}t^{-p}e^{-t}dt$发散.

(4) $\int_0^1\frac{|\ln x|^p}{(1-x^2)^q}dx$,问题集中在$0,1$

$x\to0^+$时,$\frac{|\ln x|^p}{(1-x^2)^q}=|\ln x|^p(1+o(1))^q=|\ln x|^p(1+o(1))=|\ln x|^p+o(|\ln x|^p)$

$x\to1^-$时,$\frac{|\ln x|^p}{(1-x^2)^q}\underset{\varepsilon\to0^+}{\overset{x=1-\varepsilon}{=}}\frac{|\ln(1-\varepsilon)|^p}{(1-(1-\varepsilon)^2)^q}=\frac{|\ln(1-\varepsilon)|^p}{(2\varepsilon-\varepsilon^2)^q}=\frac{1}{\varepsilon^q}\frac{\left|-\varepsilon-\frac{\varepsilon^2}{2}+o(\varepsilon)\right|^p}{(2-\varepsilon)^q}=\frac{\varepsilon^p}{2^q\varepsilon^q}\frac{\left|-1-\frac{\varepsilon}{2}+o(\varepsilon)\right|^p}{\left(1-\frac{\varepsilon}{2}\right)^q}$

$=\frac{\varepsilon^{p-q}}{2^q}\frac{\left|1+\frac{\varepsilon}{2}+o(\varepsilon)\right|^p}{\left(1-\frac{\varepsilon}{2}\right)^q}=\frac{\varepsilon^{p-q}}{2^q}\left(\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4}+o(\varepsilon^2)\right)\left(1+\frac{p\varepsilon}{2}+o(\varepsilon)\right)=\frac{\varepsilon^{p-q+1}}{2^{q+1}}+o(\varepsilon^{p-q+1})=\frac{(1-x)^{p-q+1}}{2^{q+1}}+o((1-x)^{p-q+1})$

$\frac{|\ln x|^p}{(1-x^2)^q}>0$,故只要$\int_0^1\frac{|\ln x|^p}{(1-x^2)^q}dx$在$0,1$至少一点附近发散,就有$\int_0^1\frac{|\ln x|^p}{(1-x^2)^q}dx$发散

$\int_0^1\frac{|\ln x|^p}{(1-x^2)^q}dx=\int_0^{\delta}\frac{|\ln x|^p}{(1-x^2)^q}dx+\int_{\delta}^1\frac{|\ln x|^p}{(1-x^2)^q}dx$

$x\to0^+$时,$\frac{|\ln x|^p}{(1-x^2)^q}\sim\frac{1}{|\ln x|^p}$,由(2)可知:$\int_0^{\delta}\frac{|\ln x|^p}{(1-x^2)^q}dx$收敛.

$x\to1^-$时,$\frac{|\ln x|^p}{(1-x^2)^q}\sim\frac{1}{(1-x)^{-p+q-1}}$,故$-p+q-1<1$时,$\int_{\delta}^1\frac{|\ln x|^p}{(1-x^2)^q}dx$收敛;$-p+q-1\ge1$时,$\int_{\delta}^1\frac{|\ln x|^p}{(1-x^2)^q}dx$发散.

综上:$q<p+2$时,$\int_0^1\frac{|\ln x|^p}{(1-x^2)^q}dx$收敛,否则发散.

(6) $\int_0^{\pi}\frac{(1-\cos x)^q}{x^p}dx$问题集中在$x=0$处

$x\to0^+$时,$\frac{(1-\cos x)^q}{x^p}=\frac{\left(\frac{x^2}{2}+o(x^2)\right)^q}{x^p}=\frac{x^{2q-p}}{2^q}+o(x^{2q-p})$

故$\frac{(1-\cos x)^q}{x^p}\sim\frac{1}{x^{-2q+p}}$

$-2q+p<1$时,$\int_0^{\pi}\frac{(1-\cos x)^q}{x^p}dx$收敛;$-2q+p\ge1$时,$\int_0^{\pi}\frac{(1-\cos x)^q}{x^p}dx$发散.

$3.(1)\int_0^{+\infty}\frac{\ln^q(1+x)}{x^p}dx$ 问题集中在 $0,+\infty$ 处, $\left|\frac{\ln^q(1+x)}{x^p}\right|=\frac{\ln^q(1+x)}{x^p}>0$

$\int_0^{+\infty}\frac{\ln^q(1+x)}{x^p}dx=\int_0^{\delta}\frac{\ln^q(1+x)}{x^p}dx+\int_{\delta}^{+\infty}\frac{\ln^q(1+x)}{x^p}dx$

$x\to 0^+$ 时, $\frac{\ln^q(1+x)}{x^p}=\frac{\left(x-\frac{x^2}{2}+o(x^2)\right)^q}{x^p}=x^{q-p}\left(1-\frac{x}{2}+o(x)\right)^q=x^{q-p}\left(1-\frac{q}{2}x+o(x)\right)=x^{q-p}+o(x^{q-p})$

$q-p>-1$ 时, $\int_0^{\delta}\frac{\ln^q(1+x)}{x^p}dx$ 收敛; $q-p\le -1$ 时, $\int_0^{\delta}\frac{\ln^q(1+x)}{x^p}dx$ 发散

$x\to +\infty$ 时, $\frac{\ln^q(1+x)}{x^p}=\frac{\ln^q x}{x^p}+o(1)$

$p\le 1$ 时, $\int_{\delta}^{+\infty}\frac{\ln^q(1+x)}{x^p}dx$ 发散; $p>1$ 时, $\int_{\delta}^{+\infty}\frac{\ln^q(1+x)}{x^p}dx$ 收敛

综上: $q-p>-1$ 且 $p>1$ 时, $\int_0^{+\infty}\frac{\ln^q(1+x)}{x^p}dx$ 绝对收敛; 否则发散.

$3.(2)\int_0^{+\infty}\frac{\arctan^q x}{x^p}dx$ 问题集中在 $0,+\infty$ 处, $\left|\frac{\arctan^q x}{x^p}\right|=\frac{\arctan^q x}{x^p}>0$

$\int_0^{+\infty}\frac{\arctan^q x}{x^p}dx=\int_0^{\delta}\frac{\arctan^q x}{x^p}dx+\int_{\delta}^{+\infty}\frac{\arctan^q x}{x^p}dx$

$x\to 0^+$ 时, $\frac{\arctan^q x}{x^p}=x^{q-p}+o(x^{q-p})$

$q-p>-1$ 时, $\int_0^{\delta}\frac{\arctan^q x}{x^p}dx$ 收敛; $q-p\le -1$ 时, $\int_0^{\delta}\frac{\arctan^q x}{x^p}dx$ 发散

$x\to +\infty$ 时, $\frac{\arctan^q x}{x^p}=\frac{\left(\frac{\pi}{2}-\arctan\frac{1}{x}\right)^q}{x^p}=\left(\frac{\pi}{2}\right)^q\frac{\left(1-\frac{2}{\pi}\arctan\frac{1}{x}\right)^q}{x^p}=\left(\frac{\pi}{2}\right)^q\frac{\left(1-\frac{2}{\pi}\frac{1}{x}+o\left(\frac{1}{x}\right)\right)^q}{x^p}=\left(\frac{\pi}{2}\right)^q\frac{1-\frac{2q}{\pi}\frac{1}{x}+o\left(\frac{1}{x}\right)}{x^p}$

$=\left(\frac{\pi}{2}\right)^q x^{-p}+o(x^{-p})$

$p\le 1$ 时, $\int_{\delta}^{+\infty}\frac{\arctan^q x}{x^p}dx$ 发散; $p>1$ 时, $\int_{\delta}^{+\infty}\frac{\arctan^q x}{x^p}dx$ 收敛

综上: $q-p>-1$ 且 $p>1$ 时, $\int_0^{+\infty}\frac{\arctan^q x}{x^p}dx$ 绝对收敛; 否则发散.

$(4)\int_0^{+\infty}\frac{dx}{x^p+x^q}\ (p\ge q>0)$ 问题集中在 $0,+\infty$ 处, $\left|\frac{1}{x^p+x^q}\right|=\frac{1}{x^p+x^q}>0$

①$p=q$ 时, $\int_0^{+\infty}\frac{dx}{x^p+x^q}=2\int_0^{+\infty}\frac{dx}{x^p}$ 在 $p\le 1$ 时发散, $p>1$ 时收敛.

②$p>q$ 时, $\int_0^{+\infty}\frac{dx}{x^p+x^q}=\int_0^{\delta}\frac{dx}{x^p+x^q}+\int_{\delta}^{+\infty}\frac{dx}{x^p+x^q}$

$x\to 0^+$ 时, $\frac{1}{x^p+x^q}=\frac{1}{x^q}\frac{1}{1+x^{p-q}}=-\frac{1}{x^q}(1-x^{p-q}+o(x^{p-q}))=-x^{-q}+x^{p-2q}+o(x^{p-q})=-x^{-q}+o(x^q)$

$-q>-1$ 时, $\int_0^{\delta}\frac{dx}{x^p+x^q}$ 收敛; $-q\le -1$ 时, $\int_0^{\delta}\frac{dx}{x^p+x^q}$ 发散

$x\to +\infty$ 时, $\frac{1}{x^p+x^q}=\frac{1}{x^p}\frac{1}{1+x^{q-p}}=\frac{1}{x^p}\left(1-\left(\frac{1}{x}\right)^{p-q}+o\left(\left(\frac{1}{x}\right)^{p-q}\right)\right)=\frac{1}{x^p}-\left(\frac{1}{x}\right)^{2p-q}+o\left(\left(\frac{1}{x}\right)^{2p-q}\right)=\frac{1}{x^p}+o\left(\frac{1}{x^p}\right)$

$p\ge 1$ 时, $\int_{\delta}^{+\infty}\frac{dx}{x^p+x^q}$ 收敛; $p<1$ 时, $\int_{\delta}^{+\infty}\frac{dx}{x^p+x^q}$ 发散

综上: $p=q\le 1$ 或 $p\ge 1>q$ 时, $\int_0^{+\infty}\frac{dx}{x^p+x^q}$ 绝对收敛; 否则发散.

(6) $\int_0^{+\infty} x^p \sin\frac{1}{x^q}dx$问题集中在$0,+\infty$处

$\int_0^{+\infty} x^p \sin\frac{1}{x^q}dx=\int_0^{\delta} x^p \sin\frac{1}{x^q}dx+\int_{\delta}^{+\infty} x^p \sin\frac{1}{x^q}dx$

①x^p在$(0,\delta)$单调, $\lim\limits_{x\to 0^+} x^p=0$且$\left|\int_c^{\delta}\left|\sin\frac{1}{x^q}\right|dx\right|\le\left|\int_c^{\delta}dx\right|=|\delta-c|$有界

由狄利克雷判别法: $\int_0^{\delta} x^p \sin\frac{1}{x^q}dx$绝对收敛

②$x\to+\infty$时, $x^p\left|\sin\frac{1}{x^q}\right|=x^p\sin\frac{1}{x^q}=x^p\left(\frac{1}{x^q}+o\left(\frac{1}{x^q}\right)\right)=x^{p-q}+o(x^{p-q})$

$p-q<-1$时, $\int_{\delta}^{+\infty} x^p \sin\frac{1}{x^q}dx$绝对收敛; $p-q\ge-1$时, $\int_{\delta}^{+\infty} x^p \sin\frac{1}{x^q}dx$发散.

综上: $p-q<-1$时, $\int_0^{+\infty} x^p \sin\frac{1}{x^q}dx$绝对收敛; $p-q\ge-1$时, $\int_0^{+\infty} x^p \sin\frac{1}{x^q}dx$发散.

(8) $\int_0^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^p}dx=\int_0^{1}\frac{\sin\left(x+\frac{1}{x}\right)}{x^p}dx+\int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^p}dx=\int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^p}dx+\int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^{2-p}}dx$

只需对于$q\in\mathbb{R}$, 考察$\int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^q}dx$的敛散性

$q>0$时, 由狄利克雷判别法: $\int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^q}dx$收敛(这个证明非常难, 我们放在后面)

①$q>1$时, $\int_1^{+\infty}\frac{\left|\sin\left(x+\frac{1}{x}\right)\right|}{x^q}dx<\int_1^{+\infty}\frac{1}{x^q}dx<\infty$, 且$\int_1^{a}\frac{\left|\sin\left(x+\frac{1}{x}\right)\right|}{x^q}dx$随$a\uparrow$而$\uparrow$

由单调有界原理: $\int_1^{+\infty}\frac{\left|\sin\left(x+\frac{1}{x}\right)\right|}{x^q}dx$收敛

②$0<q\le1$时, $\int_1^{+\infty}\frac{\left|\sin\left(x+\frac{1}{x}\right)\right|}{x^q}dx\ge\sum_{n=1}^{\infty}\int_{2n\pi}^{2(n+1)\pi}\frac{\left|\sin\left(x+\frac{1}{x}\right)\right|}{x^q}dx\ge\sum_{n=1}^{\infty}\int_{2n\pi+\frac{\pi}{4}}^{2n\pi+\frac{\pi}{2}}\frac{\left|\sin\left(x+\frac{1}{x}\right)\right|}{x^q}dx\ge\frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}\int_{2n\pi+\frac{\pi}{4}}^{2n\pi+\frac{\pi}{2}}\frac{1}{x^q}dx$

$\ge\frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}\frac{\frac{\pi}{4}}{\left(2n\pi+\frac{\pi}{2}\right)^q}=\frac{\sqrt{2}}{2^{q+3}\pi^{q-1}}\sum_{n=1}^{\infty}\frac{1}{\left(n+\frac{1}{4}\right)^q}\to+\infty$, 故$\int_1^{+\infty}\frac{\left|\sin\left(x+\frac{1}{x}\right)\right|}{x^q}dx$发散

$q\le0$时, 取$x_n=2n\pi+\frac{\pi}{4}, y_n=2n\pi+\frac{\pi}{2}$, 且在$n$充分大时, 有$x_n+\frac{1}{x_n}, y_n+\frac{1}{y_n}\in\left(2n\pi+\frac{\pi}{4}, 2n\pi+\frac{3\pi}{4}\right)$

$\left|\int_{x_n}^{y_n}\frac{\sin\left(x+\frac{1}{x}\right)}{x^q}dx\right|=\left|\sin\left(\zeta+\frac{1}{\zeta}\right)\int_{x_n}^{y_n}\frac{1}{x^q}dx\right|=\left|\sin\left(\zeta+\frac{1}{\zeta}\right)\right|\left|\int_{x_n}^{y_n}\frac{1}{x^q}dx\right|\ge\frac{\sqrt{2}}{2}x_n^{-q}(y_n-x_n)=\frac{\sqrt{2}\pi}{8}x_n^{-q}\ge\frac{\sqrt{2}\pi}{8}>0$

其中$\zeta\in[x_n,y_n], \zeta+\frac{1}{\zeta}\in\left[x_n+\frac{1}{x_n}, y_n+\frac{1}{y_n}\right]\subset\left(2n\pi+\frac{\pi}{4}, 2n\pi+\frac{3\pi}{4}\right)$

由柯西收敛准则: $\int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^q}dx$发散

综上: $q>1$时, $\int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^q}dx$绝对收敛; $0<q\le1$时, $\int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^q}dx$条件收敛; $q\le0$时, $\int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^q}dx$发散

$$\begin{cases} p>1\text{时}, \int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^p}dx\text{绝对收敛}; 0<p\le1\text{时}, \int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^p}dx\text{条件收敛}; p\le0\text{时}, \int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^p}dx\text{发散} \\ p<1\text{时}, \int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^{2-p}}dx\text{绝对收敛}; 1\le p<2\text{时}, \int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^{2-p}}dx\text{条件收敛}; p\ge2\text{时}, \int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^{2-p}}dx\text{发散} \end{cases}$$

故$\int_0^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^p}dx$在$0<p<2$时条件收敛, $p\ge2$时发散.

Step 1 :我们试图证明: $\int_1^a \sin\left(x+\frac{1}{x}\right)dx$有界, $\forall a>1$

由于有界闭区间上的常义黎曼可积函数有界

只需证: $\int_{2\pi}^a \sin\left(x+\frac{1}{x}\right)dx$有界, $\forall a>2\pi$

由于有界闭区间上的常义黎曼可积函数有界,且任意a必然落在某个n形成的$[2n\pi,2(n+1)\pi)$中

只需证: $\int_{2\pi}^{2n\pi} \sin\left(x+\frac{1}{x}\right)dx$有界, $\forall n\in\mathbb{N}^*$

$$\int_{2\pi}^{2n\pi}\sin\left(x+\frac{1}{x}\right)dx=\int_{2\pi}^{2n\pi}\sin x\cos\frac{1}{x}dx+\int_{2\pi}^{2n\pi}\sin\frac{1}{x}\cos x dx$$

①我们断言$\int_{2\pi}^{2n\pi}\sin x\cos\frac{1}{x}dx$有界

$$\left|\int_{2\pi}^{2n\pi}\sin x\cos\frac{1}{x}dx\right|=\left|\sum_{k=1}^{n-1}\left(\int_{2k\pi}^{2k\pi+\pi}\sin x\cos\frac{1}{x}dx+\int_{2k\pi+\pi}^{2k\pi+2\pi}\sin x\cos\frac{1}{x}dx\right)\right|$$

$$=\left|\sum_{k=1}^{n-1}\left(\int_0^\pi\sin x\left(\cos\frac{1}{2k\pi+x}-\cos\frac{1}{2k\pi+\pi+x}\right)dx\right)\right|$$

$$=\left|\sum_{k=1}^{n-1}\int_0^\pi\sin x(-\sin\theta_k(x))\left(\frac{1}{2k\pi+x}-\frac{1}{2k\pi+\pi+x}\right)dx\right|,\text{其中}\theta_k(x)\text{介于}2k\pi+x\text{和}2k\pi+\pi+x\text{之间}$$

$$\le\sum_{k=1}^{n-1}\int_0^\pi\sin x|\sin\theta_k(x)|\frac{\pi}{(2k\pi+\pi+x)(2k\pi+x)}dx\le\pi\sum_{k=1}^{n-1}\int_0^\pi\frac{\sin x}{(2k\pi+\pi+x)(2k\pi+x)}dx$$

$$\le\pi\sum_{k=1}^{n-1}\int_0^\pi\frac{\sin x}{(2k\pi)^2}dx=\frac{1}{4\pi}\sum_{k=1}^{n-1}\frac{1}{k^2}\int_0^\pi\sin x dx\le\frac{1}{4\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^\pi\sin x dx=\frac{1}{4\pi}\cdot\frac{\pi^2}{6}\cdot2=\frac{\pi}{12}\text{有界}$$

②我们断言$\int_{2\pi}^{2n\pi}\sin\frac{1}{x}\cos x dx$有界

$$\left|\int_{2\pi}^{2n\pi}\sin\frac{1}{x}\cos x dx\right|=\left|\sum_{k=1}^{n-1}\left(\int_{2k\pi}^{2k\pi+\pi}\sin\frac{1}{x}\cos x dx+\int_{2k\pi+\pi}^{2k\pi+2\pi}\sin\frac{1}{x}\cos x dx\right)\right|$$

$$=\left|\sum_{k=1}^{n-1}\left(\int_0^\pi\cos x\left(\sin\frac{1}{2k\pi+x}-\sin\frac{1}{2k\pi+\pi+x}\right)dx\right)\right|$$

$$\le\left|\sum_{k=1}^{n-1}\int_0^\pi\cos x(\cos\theta_k(x))\left(\frac{1}{2k\pi+x}-\frac{1}{2k\pi+\pi+x}\right)dx\right|,\text{其中}\theta_k(x)\text{介于}2k\pi+x\text{和}2k\pi+\pi+x\text{之间}$$

$$\le\sum_{k=1}^{n-1}\int_0^\pi|\cos x||\cos\theta_k(x)|\frac{\pi}{(2k\pi+\pi+x)(2k\pi+x)}dx\le\pi\sum_{k=1}^{n-1}\int_0^\pi\frac{|\cos x|}{(2k\pi+\pi+x)(2k\pi+x)}dx$$

$$\le\pi\sum_{k=1}^{n-1}\int_0^\pi\frac{|\cos x|}{(2k\pi)^2}dx=\frac{1}{4\pi}\sum_{k=1}^{n-1}\frac{1}{k^2}\int_0^\pi|\cos x|dx\le\frac{1}{4\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^\pi|\cos x|dx=\frac{1}{4\pi}\cdot\frac{\pi^2}{6}\cdot2=\frac{\pi}{12}\text{有界}$$

综上: $\left|\int_{2\pi}^{2n\pi}\sin\left(x+\frac{1}{x}\right)dx\right|\le\left|\int_{2\pi}^{2n\pi}\sin x\cos\frac{1}{x}dx\right|+\left|\int_{2\pi}^{2n\pi}\sin\frac{1}{x}\cos x dx\right|\le\frac{\pi}{6}$有界

故$\int_1^a\sin\left(x+\frac{1}{x}\right)dx$有界, $\forall a>1$

Step 2 :这点结合$\frac{1}{x^q}(q>0)$递减趋于0,即可由狄利克雷判别法得出: $\int_1^{+\infty}\frac{\sin\left(x+\frac{1}{x}\right)}{x^q}dx$收敛

5.(1) $1-x^2\le 1-x^4\le 2-2x^2(0<x<1)\Rightarrow \frac{\pi}{2\sqrt{2}}=\int_0^1\frac{dx}{\sqrt{2-2x^2}}<\int_0^1\frac{dx}{\sqrt{1-x^4}}<\int_0^1\frac{dx}{\sqrt{1-x^2}}=\frac{\pi}{2}$

(2) $\frac{1-\frac{x^2}{2}}{\sqrt{1-x^2}}\le\frac{\cos x}{\sqrt{1-x^2}}\le\frac{1}{\sqrt{1-x^2}}(0<x<1)\Rightarrow\frac{3\pi}{8}=\int_0^1\frac{1-\frac{x^2}{2}}{\sqrt{1-x^2}}dx<\int_0^1\frac{\cos x}{\sqrt{1-x^2}}dx<\int_0^1\frac{1}{\sqrt{1-x^2}}dx=\frac{\pi}{2}$

(4)法一: $\int_0^{+\infty}e^{-x^2}dx=\left[\left(\int_0^{+\infty}e^{-x^2}dx\right)^2\right]^{\frac{1}{2}}=\left(\int_0^{+\infty}e^{-x^2}dx\int_0^{+\infty}e^{-y^2}dy\right)^{\frac{1}{2}}=\left(\int_0^{+\infty}\int_0^{+\infty}e^{-(x^2+y^2)}dx\,dy\right)^{\frac{1}{2}}$

$=\left(\int_0^{\frac{\pi}{2}}d\theta\int_0^{+\infty}re^{-r^2}dr\right)^{\frac{1}{2}}=\left(\int_0^{\frac{\pi}{2}}d\theta\int_0^{+\infty}re^{-r^2}dr\right)^{\frac{1}{2}}=\frac{\sqrt{\pi}}{2}\in\left(\frac{1}{2}\left(1-\frac{1}{e}\right),1+\frac{1}{2e}\right)$

法二: $\int_0^{+\infty}e^{-x^2}dx=\int_0^1e^{-x^2}dx+\int_1^{+\infty}e^{-x^2}dx<\int_0^1 1dx+\int_1^{+\infty}xe^{-x^2}dx=\int_0^1 1dx+\int_1^{+\infty}\frac{1}{2}e^{-x^2}dx^2=1+\frac{1}{2e}$

$\int_0^{+\infty}e^{-x^2}dx>\int_0^1e^{-x^2}dx>\int_0^1xe^{-x^2}dx=\frac{1}{2}\int_0^1e^{-x^2}dx^2=\frac{1}{2}\left(1-\frac{1}{e}\right)$，得证!

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1.(1) $\int_0^1 \frac{x^3}{\sqrt{1-x^2}}dx = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos x} d\sin x = \int_0^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}$

(2) $\int_{\sqrt{2}}^{+\infty}\frac{dx}{(x-1)\sqrt{x^2-2}} \overset{t=\sqrt{x^2-2}}{\underset{x=\sqrt{t^2+2}}{=}} \int_0^{+\infty}\frac{d\sqrt{t^2+2}}{(\sqrt{t^2+2}-1)t} = \int_0^{+\infty}\frac{dt}{(\sqrt{t^2+2}-1)\sqrt{t^2+2}} = \int_0^{+\infty}\frac{\sqrt{t^2+2}+1}{(t^2+1)\sqrt{t^2+2}}dt$

$= \int_0^{+\infty}\frac{1}{t^2+1}dt + \int_0^{+\infty}\frac{1}{(t^2+1)\sqrt{t^2+2}}dt$

$\int_0^{+\infty}\frac{1}{t^2+1}dt = \frac{\pi}{2}$

$\int_0^{+\infty}\frac{1}{(t^2+1)\sqrt{t^2+2}}dt \overset{t=\tan x}{=} \int_0^{\frac{\pi}{2}}\frac{1}{(\tan^2 x+1)\sqrt{\tan^2 x+2}}d\tan x = \int_0^{\frac{\pi}{2}}\frac{1}{\sec^2 x\sqrt{\tan^2 x+2}}\sec^2 x dx$

$= \int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{\tan^2 x+2}}dx = \int_0^{\frac{\pi}{2}}\frac{\cos x}{\sqrt{2\cos^2 x+\sin^2 x}}dx = \int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{2-\sin^2 x}}d\sin x = \arcsin\frac{\sin x}{\sqrt{2}}\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

$\Rightarrow \int_{\sqrt{2}}^{+\infty}\frac{dx}{(x-1)\sqrt{x^2-2}} = \int_0^{+\infty}\frac{1}{t^2+1}dt + \int_0^{+\infty}\frac{1}{(t^2+1)\sqrt{t^2+2}}dt = \frac{\pi}{2}+\frac{\pi}{4} = \frac{3\pi}{4}$

(4) $\int_0^{+\infty}\frac{dx}{\sqrt{x}(1+\sqrt[4]{x})^3} \overset{x=t^4}{=} \int_0^{+\infty}\frac{4t^3 dt}{t^2(1+t)^3} = 4\int_0^{+\infty}\frac{t dt}{(1+t)^3} = 4\int_0^{+\infty}\left[\frac{1}{(1+t)^2}-\frac{1}{(1+t)^3}\right]dt$

$\int_0^{+\infty}\frac{1}{(1+t)^2}dt = -\frac{1}{1+t}\Big|_0^{+\infty} = 1$

$\int_0^{+\infty}\frac{1}{(1+t)^3}dt = -\frac{2}{(1+t)^2}\Big|_0^{+\infty} = -2$

$\Rightarrow \int_0^{+\infty}\frac{dx}{\sqrt{x}(1+\sqrt[4]{x})^3} = 4\int_0^{+\infty}\left[\frac{1}{(1+t)^2}-\frac{1}{(1+t)^3}\right]dt = 4$

(6) $\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt[3]{\tan x}} \overset{t=\sqrt[3]{\tan x}}{\underset{x=\arctan t^3}{=}} \int_0^{+\infty}\frac{d\arctan t^3}{t} = \int_0^{+\infty}\frac{3t^2 dt}{t(1+t^6)} = \int_0^{+\infty}\frac{3t dt}{1+t^6} = \frac{3}{2}\int_0^{+\infty}\frac{dt^2}{1+t^6} = \frac{3}{2}\int_0^{+\infty}\frac{dx}{1+x^3}$

$\int_0^{+\infty}\frac{dx}{1+x^3} = -\int_0^{+\infty}\frac{d\frac{1}{x}}{1+\left(\frac{1}{x}\right)^3} = \int_0^{+\infty}\frac{x dx}{1+x^3} = \frac{1}{2}\int_0^{+\infty}\frac{(1+x)dx}{1+x^3} = \frac{1}{2}\int_0^{+\infty}\frac{dx}{x^2-x+1} = \frac{1}{2}\int_0^{+\infty}\frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$

$= \frac{1}{2}\int_0^{+\infty} d\frac{1}{\sqrt{\frac{3}{4}}}\arctan\frac{x-\frac{1}{2}}{\sqrt{\frac{3}{4}}} = \frac{2\pi}{3\sqrt{3}}$

$\Rightarrow \int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt[3]{\tan x}} = \frac{3}{2}\int_0^{+\infty}\frac{dx}{1+x^3} = \frac{\pi}{\sqrt{3}}$

(8) $\int_0^{+\infty}\frac{x\arctan\sqrt{x}}{(1+x^2)^2}dx = -\int_0^{+\infty}\frac{\frac{1}{x}\arctan\sqrt{\frac{1}{x}}}{\left(1+\left(\frac{1}{x}\right)^2\right)^2}d\left(\frac{1}{x}\right) = \int_0^{+\infty}\frac{x\left(\frac{\pi}{2}-\arctan\sqrt{x}\right)}{(1+x^2)^2}dx = \frac{\pi}{4}\int_0^{+\infty}\frac{x}{(1+x^2)^2}dx$

$= \frac{\pi}{8}\int_0^{+\infty}\frac{1}{(1+x^2)^2}d(1+x^2) = \frac{\pi}{8}$

(10) $\int_0^{+\infty}\frac{\arctan e^{\frac{x}{2}}}{e^{\frac{x}{2}}(1+e^x)}dx = \int_0^{+\infty}\frac{e^{\frac{x}{2}}\arctan e^{\frac{x}{2}}}{e^x(1+e^x)}dx = 2\int_0^{+\infty}\frac{\arctan e^{\frac{x}{2}}}{e^x(1+e^x)}de^{\frac{x}{2}} = 2\int_1^{+\infty}\frac{\arctan x}{x^2(1+x^2)}dx$

$= -\int_1^{+\infty}\arctan x d\left(\frac{1}{x}+\arctan x\right) = -\arctan x\left(\frac{1}{x}+\arctan x\right)\Big|_1^{+\infty} + \int_1^{+\infty}\left(\frac{1}{x}+\arctan x\right)d\arctan x$

$= -\frac{\pi^2}{4}+\frac{\pi}{4}\left(1+\frac{\pi}{4}\right)+\int_1^{+\infty}\frac{1}{x(1+x^2)}dx + \int_1^{+\infty}\arctan x d\arctan x = -\frac{3\pi^2}{16}+\frac{\pi}{4}+\int_1^{+\infty}\frac{x}{x^2(1+x^2)}dx+\frac{1}{2}\int_1^{+\infty}d(\arctan x)^2$

$= -\frac{3\pi^2}{16}+\frac{\pi}{4}+\frac{1}{2}\int_1^{+\infty}\frac{1}{x^2(1+x^2)}dx^2+\frac{1}{2}\int_1^{+\infty}d(\arctan x)^2 = -\frac{3\pi^2}{16}+\frac{\pi}{4}+\frac{1}{2}\int_1^{+\infty}\frac{1}{x(1+x)}dx+\frac{1}{2}\left(\left(\frac{\pi}{2}\right)^2-\left(\frac{\pi}{4}\right)^2\right)$

$= -\frac{3\pi^2}{32}+\frac{\pi}{4}+\frac{1}{2}\ln 2 = 0.206696$

$$4.\int_{-\infty}^{+\infty} e^{-ax^2-\frac{b}{x^2}}dx = 2\int_0^{+\infty} e^{-ax^2-\frac{b}{x^2}}dx \overset{x=\sqrt{\frac{b}{a}}\frac{1}{y}}{=} -2\sqrt{\frac{b}{a}}\int_0^{+\infty} e^{-a\left(\sqrt{\frac{b}{a}}\frac{1}{y}\right)^2-\frac{b}{\left(\sqrt{\frac{b}{a}}\frac{1}{y}\right)^2}} d\frac{1}{y} = 2\sqrt{\frac{b}{a}}\int_0^{+\infty}\frac{1}{y^2}e^{-ay^2-\frac{b}{y^2}}dy$$

$$=\int_0^{+\infty}\left(1+\sqrt{\frac{b}{a}}\frac{1}{x^2}\right)e^{-ax^2-\frac{b}{x^2}}dx = \frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-ax^2-\frac{b}{x^2}} d\left(\sqrt{a}\,x-\frac{\sqrt{b}}{x}\right) = \frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-\left(\sqrt{a}\,x-\frac{\sqrt{b}}{x}\right)^2-2\sqrt{ab}} d\left(\sqrt{a}\,x-\frac{\sqrt{b}}{x}\right)$$

$$=\frac{e^{-2\sqrt{ab}}}{\sqrt{a}}\int_0^{+\infty} e^{-\left(\sqrt{a}\,x-\frac{\sqrt{b}}{x}\right)^2} d\left(\sqrt{a}\,x-\frac{\sqrt{b}}{x}\right) = \frac{\sqrt{\pi}\,e^{-2\sqrt{ab}}}{\sqrt{a}}$$

$$7.(2)\int_0^{4\pi} x\sin x\,\mathrm{sgn}(\cos x)dx = \int_0^{\frac{\pi}{2}} x\sin x dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x\sin x dx + \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x\sin x dx - \int_{\frac{5\pi}{2}}^{\frac{7\pi}{2}} x\sin x dx + \int_{\frac{7\pi}{2}}^{4\pi} x\sin x dx$$

$$=-\int_0^{\frac{\pi}{2}} xd\cos x + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} xd\cos x - \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} xd\cos x + \int_{\frac{5\pi}{2}}^{\frac{7\pi}{2}} xd\cos x - \int_{\frac{7\pi}{2}}^{4\pi} xd\cos x$$

$$=\int_0^{\frac{\pi}{2}}\cos x dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos x dx + \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}}\cos x dx - \int_{\frac{5\pi}{2}}^{\frac{7\pi}{2}}\cos x dx + \int_{\frac{7\pi}{2}}^{4\pi}\cos x dx - 4\pi$$

$$=1+2+2+2+1-4\pi=8-4\pi$$

$$(3)\int_{-1}^{2}\frac{\mathrm{sgn}x}{\sqrt{|x|}}dx \overset{\text{由对称性}}{=} \int_1^2\frac{1}{\sqrt{x}}dx = 2\int_1^2 d\sqrt{x} = 2\sqrt{2}-2$$

8.**Pf:** (1)法一：$F(x)$在(a,b)广义可积.

由于区间长度趋于0时，积分值也趋于0.

故$F(x)$在(a,b)连续

法二：①对于一个区间(α,β)，f在(α,β)内无瑕点，f在(α,β)内广义可积.

f只可能在α,β处不存在.

故对于任意$n\in\mathbb{N},n>\dfrac{2}{\beta-\alpha}$,$f$在$\left[\alpha+\dfrac{1}{n},\beta-\dfrac{1}{n}\right]$内常义可积.

故f在$\left[\alpha+\dfrac{1}{n},\beta-\dfrac{1}{n}\right]$内有界,记$\sup\limits_{x\in\left[\alpha+\frac{1}{n},\beta-\frac{1}{n}\right]}|f(x)|=M_n$

故$|F(y)-F(x)|=\left|\int_x^y f(t)dt\right|\le M_n|y-x|$，$F$在$\left[\alpha+\dfrac{1}{n},\beta-\dfrac{1}{n}\right]$上*Lipschitz*连续.

故F在$(\alpha,\beta)=\bigcup\limits_{n=\left[\frac{2}{\beta-\alpha}\right]+1}^{\infty}\left[\alpha+\dfrac{1}{n},\beta-\dfrac{1}{n}\right]$上连续.

②若(a,b)内有个瑕点c,证明F在c处连续.

由于f在(a,b)内广义可积

故$\lim\limits_{\varepsilon\to0^+}\int_a^{c+\varepsilon}f(t)dt=\int_a^b f(t)dt-\lim\limits_{\varepsilon\to0^+}\int_{c+\varepsilon}^b f(t)dt=\int_a^b f(t)dt-\int_c^b f(t)dt=\int_a^c f(t)dt=\lim\limits_{\varepsilon\to0^+}\int_a^{c-\varepsilon}f(t)dt$

故$\lim\limits_{\varepsilon\to0^+}F(c+\varepsilon)=\lim\limits_{\varepsilon\to0^+}F(c-\varepsilon)$,$F$在$c$处连续.

综上:$F(x)$在(a,b)上连续.

(2)若$f(x)$在x_0处连续，$\forall\varepsilon>0$, $\exists\delta>0$, $s.t.|f(x)-f(x_0)|<\varepsilon,\forall x\in(x_0-\delta,x_0+\delta)$

令$h<\delta$, $\lim\limits_{h\to0^+}\dfrac{F(x_0+h)-F(x_0)}{h}=\lim\limits_{h\to0^+}\dfrac{\int_{x_0}^{x_0+h}f(t)dt}{h}\le\lim\limits_{h\to0^+}\dfrac{\int_{x_0}^{x_0+h}(f(x_0)+\varepsilon)dt}{h}=f(x_0)+\varepsilon$

$$\lim_{h\to0^+}\frac{F(x_0+h)-F(x_0)}{h}=\lim_{h\to0^+}\frac{\int_{x_0}^{x_0+h}f(t)dt}{h}\ge\lim_{h\to0^+}\frac{\int_{x_0}^{x_0+h}(f(x_0)-\varepsilon)dt}{h}=f(x_0)-\varepsilon$$

$$\lim_{h\to0^+}\frac{F(x_0)-F(x_0-h)}{h}=\lim_{h\to0^+}\frac{\int_{x_0-h}^{x_0}f(t)dt}{h}\le\lim_{h\to0^+}\frac{\int_{x_0-h}^{x_0}(f(x_0)+\varepsilon)dt}{h}=f(x_0)+\varepsilon$$

$$\lim_{h\to0^+}\frac{F(x_0)-F(x_0-h)}{h}=\lim_{h\to0^+}\frac{\int_{x_0-h}^{x_0}f(t)dt}{h}\ge\lim_{h\to0^+}\frac{\int_{x_0-h}^{x_0}(f(x_0)-\varepsilon)dt}{h}=f(x_0)-\varepsilon$$

由ε任意性: $\lim\limits_{h\to0^+}\dfrac{F(x_0)-F(x_0-h)}{h}=f(x_0)=\lim\limits_{h\to0^+}\dfrac{F(x_0+h)-F(x_0)}{h}$

故F在x_0处可导,$F'(x_0)=f(x_0)$