7. 对二元函数 $u=f(x,y)$, 令 $x=r\cos\theta$, $y=r\sin\theta$.

(1) 求 $\dfrac{\partial u}{\partial r}$ 和 $\dfrac{\partial u}{\partial \theta}$;

(2) 证明 $\nabla u=\dfrac{\partial u}{\partial r}\boldsymbol{e}_r+\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}\boldsymbol{e}_\theta$, 其中, $\boldsymbol{e}_r$ 表示极径 r 方向的单位向量, $\boldsymbol{e}_\theta$ 表示极角 θ 方向的单位向量, 即 $\boldsymbol{e}_r=\dfrac{(x,y)}{r}$, $\boldsymbol{e}_\theta=\dfrac{(-y,x)}{r}$;

(3) 证明 $|\nabla u|^2=\left(\dfrac{\partial u}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial \theta}\right)^2$.

9. 定义在环形区域 $a<|x|<b$(其中 $x\in\mathbf{R}^m$) 上的函数 $f(x)$ 称为**球对称函数**或**径向函数**, 如果存在定义于区间 (a,b) 上的函数 φ 使成立 $f(x)=\varphi(r)$, 其中 $r=|x|\in(a,b)$. 假设 φ 是 (a,b) 上的可微函数, 对径向函数 $f(x)=\varphi(r)$ 证明以下结论:

(1) $f(x)$ 是环形区域 $a<|x|<b$ 上的可微函数, 且

$$\nabla f(x)=\varphi'(r)\frac{x}{r},\quad \forall r=|x|\in(a,b);$$

(2) $\dfrac{\partial f}{\partial r}(x)=\varphi'(r)$, 这是 $\dfrac{\partial f}{\partial r}(x)$ 表示 $f(x)$ 关于径向 $\boldsymbol{e}_r=\dfrac{x}{r}$ 的方向导数;

(3) 对任意 $a<|x_0|<b$ 和任意垂直于点 x_0 的向径的单位向量 $\boldsymbol{l}$ ($\boldsymbol{l}\cdot x_0=0$), 有

$$\frac{\partial f}{\partial \boldsymbol{l}}(x_0)=0.$$

10. 证明: 如果把隐函数定理中最后的条件 "每个偏导数 $F_{x_i}(x,y)$ $(i=1,2,\cdots,m)$ 都在 Ω 上连续" 减弱为 $F(x,y)$ 对每个固定的 y 都关于 x 可微, 则相应的隐函数 $f(x)$ 在 $B(x_0,r)$ 上可微.

11. 设 f 是一元可微函数, F 是二元可微函数. 验证由所给定的隐函数方程确定的隐函数满足相应的等式:

(1) $a\dfrac{\partial u}{\partial x}+b\dfrac{\partial u}{\partial y}=1$, 其中 u 为由方程 $F(x-au,y-bu)=0$ 确定的隐函数;

(4) $(x^2-y^2-u^2)\dfrac{\partial u}{\partial x}+2xy\dfrac{\partial u}{\partial y}=2xu$, 其中 u 为由方程 $x^2+y^2+u^2=yf\left(\dfrac{u}{y}\right)$ 确定的隐函数;

12. 设 $F(x,y,z)$ 是开集 $\Omega\subseteq\mathbf{R}^3$ 中的可微函数, 且对任意 $(x,y,z)\in\Omega$ 都有 $F_x(x,y,z)\neq 0$, $F_y(x,y,z)\neq 0$ 和 $F_z(x,y,z)\neq 0$. 令 $x=x(y,z)$, $y=y(x,z)$ 和 $z=z(x,y)$ 为由方程 $F(x,y,z)=0$ 确定的隐函数. 证明:

$$\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x}=-1,\quad \frac{\partial x}{\partial z}\cdot\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial x}=-1.$$

7.(1) $\dfrac{\partial u}{\partial r}=\dfrac{\partial f(x,y)}{\partial r}=\dfrac{\partial f(x,y)}{\partial x}\dfrac{\partial x}{\partial r}+\dfrac{\partial f(x,y)}{\partial y}\dfrac{\partial y}{\partial r}=D_1 f(x,y)\cos\theta+D_2 f(x,y)\sin\theta$

$\dfrac{\partial u}{\partial \theta}=\dfrac{\partial f(x,y)}{\partial \theta}=\dfrac{\partial f(x,y)}{\partial x}\dfrac{\partial x}{\partial \theta}+\dfrac{\partial f(x,y)}{\partial y}\dfrac{\partial y}{\partial \theta}=-D_1 f(x,y)r\sin\theta+D_2 f(x,y)r\cos\theta$

7.(2) Check: $\nabla u=\dfrac{\partial u}{\partial r}\boldsymbol{e}_r+\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}\boldsymbol{e}_\theta=\dfrac{\partial u}{\partial r}\dfrac{(x,y)}{r}+\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}\dfrac{(-y,x)}{r}$

$\nabla u=\nabla f(x,y)=(D_1 f(x,y),D_2 f(x,y))$

$\dfrac{\partial u}{\partial r}\dfrac{(x,y)}{r}+\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}\dfrac{(-y,x)}{r}=\dfrac{\partial u}{\partial r}\dfrac{(r\cos\theta,r\sin\theta)}{r}+\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}\dfrac{(-r\sin\theta,r\cos\theta)}{r}$

$=(D_1 f(x,y)\cos\theta+D_2 f(x,y)\sin\theta)\dfrac{(r\cos\theta,r\sin\theta)}{r}+\dfrac{1}{r}(-D_1 f(x,y)r\sin\theta+D_2 f(x,y)r\cos\theta)\dfrac{(-r\sin\theta,r\cos\theta)}{r}$

$=(D_1 f(x,y)\cos^2\theta+D_2 f(x,y)\sin\theta\cos\theta+D_1 f(x,y)\sin^2\theta-D_2 f(x,y)\sin\theta\cos\theta,D_1 f(x,y)\sin\theta\cos\theta+D_2 f(x,y)\sin^2\theta-D_1 f(x,y)\sin\theta\cos\theta+D_2 f(x,y)\cos^2\theta)$

$=(D_1 f(x,y),D_2 f(x,y))=\nabla u.\square$

7.(3) Check: $|\nabla u|^2=\left(\dfrac{\partial u}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial \theta}\right)^2$

$|\nabla u|^2=|(D_1 f(x,y),D_2 f(x,y))|^2=[D_1 f(x,y)]^2+[D_2 f(x,y)]^2$

$\left(\dfrac{\partial u}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial \theta}\right)^2=(D_1 f(x,y)\cos\theta+D_2 f(x,y)\sin\theta)^2+\dfrac{1}{r^2}(-D_1 f(x,y)r\sin\theta+D_2 f(x,y)r\cos\theta)^2$

$=[D_1 f(x,y)]^2+[D_2 f(x,y)]^2=|\nabla u|^2.\square$、

9. (1) 对于 $a<|\boldsymbol{x}|<b$, $\boldsymbol{h}\in\mathbb{R}^m$ 趋于 0, 有

$$\left|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})-\varphi'(r)\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\cdot\boldsymbol{h}\right|=\left|\varphi(|\boldsymbol{x}+\boldsymbol{h}|)-\varphi(|\boldsymbol{x}|)-\varphi'(r)\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\cdot\boldsymbol{h}\right|$$

$$\leq|\varphi(|\boldsymbol{x}+\boldsymbol{h}|)-\varphi(|\boldsymbol{x}|)-\varphi'(r)(|\boldsymbol{x}+\boldsymbol{h}|-|\boldsymbol{x}|)|+\left|\varphi'(r)(|\boldsymbol{x}+\boldsymbol{h}|-|\boldsymbol{x}|)-\varphi'(r)\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\cdot\boldsymbol{h}\right|$$

$$\leq o(|\boldsymbol{x}+\boldsymbol{h}|-|\boldsymbol{x}|)+|\varphi'(r)|\left||\boldsymbol{x}+\boldsymbol{h}|-|\boldsymbol{x}|-\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\cdot\boldsymbol{h}\right|$$

$$=o(|\boldsymbol{h}|)+|\varphi'(r)|\left||\boldsymbol{x}+\boldsymbol{h}|-|\boldsymbol{x}|-\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\cdot\boldsymbol{h}\right|$$

只需验证 $\left||\boldsymbol{x}+\boldsymbol{h}|-|\boldsymbol{x}|-\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\cdot\boldsymbol{h}\right|=o(|\boldsymbol{h}|)$, 即 $\lim\limits_{|\boldsymbol{h}|\to 0}\dfrac{\left||\boldsymbol{x}+\boldsymbol{h}|-|\boldsymbol{x}|-\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\cdot\boldsymbol{h}\right|}{|\boldsymbol{h}|}=0$

记 $\boldsymbol{x}=(x_1,\cdots,x_m)$, $\boldsymbol{h}=(h_1,\cdots,h_m)$

于是 $$\frac{\left||\boldsymbol{x}+\boldsymbol{h}|-|\boldsymbol{x}|-\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\cdot\boldsymbol{h}\right|}{|\boldsymbol{h}|}=\left|\frac{\sqrt{(x_1+h_1)^2+\cdots+(x_m+h_m)^2}-\sqrt{x_1^2+\cdots+x_m^2}-\frac{x_1h_1+\cdots+x_mh_m}{\sqrt{x_1^2+\cdots+x_m^2}}}{\sqrt{h_1^2+\cdots+h_m^2}}\right|$$

$$=\left|\frac{\frac{2x_1h_1+\cdots+2x_mh_m+h_1^2+\cdots+h_m^2}{\sqrt{(x_1+h_1)^2+\cdots+(x_m+h_m)^2}+\sqrt{x_1^2+\cdots+x_m^2}}-\frac{x_1h_1+\cdots+x_mh_m}{\sqrt{x_1^2+\cdots+x_m^2}}}{\sqrt{h_1^2+\cdots+h_m^2}}\right|$$

$$\leq\left|\frac{(2x_1h_1+\cdots+2x_mh_m+h_1^2+\cdots+h_m^2)\sqrt{x_1^2+\cdots+x_m^2}-(x_1h_1+\cdots+x_mh_m)\left[\sqrt{(x_1+h_1)^2+\cdots+(x_m+h_m)^2}+\sqrt{x_1^2+\cdots+x_m^2}\right]}{\sqrt{h_1^2+\cdots+h_m^2}\sqrt{x_1^2+\cdots+x_m^2}\left[\sqrt{(x_1+h_1)^2+\cdots+(x_m+h_m)^2}+\sqrt{x_1^2+\cdots+x_m^2}\right]}\right|$$

$$\leq\left|\frac{(2x_1h_1+\cdots+2x_mh_m+h_1^2+\cdots+h_m^2)\sqrt{x_1^2+\cdots+x_m^2}-(x_1h_1+\cdots+x_mh_m)\left(2\sqrt{x_1^2+\cdots+x_m^2}+\sqrt{h_1^2+\cdots+h_m^2}\right)}{2\sqrt{h_1^2+\cdots+h_m^2}(x_1^2+\cdots+x_m^2)}\right|$$

$$=\left|\frac{(h_1^2+\cdots+h_m^2)\sqrt{x_1^2+\cdots+x_m^2}-(x_1h_1+\cdots+x_mh_m)\sqrt{h_1^2+\cdots+h_m^2}}{2\sqrt{h_1^2+\cdots+h_m^2}(x_1^2+\cdots+x_m^2)}\right|$$

$$=\left|\frac{\sqrt{h_1^2+\cdots+h_m^2}\sqrt{x_1^2+\cdots+x_m^2}-(x_1h_1+\cdots+x_mh_m)}{2(x_1^2+\cdots+x_m^2)}\right|$$

$$\leq\left|\frac{\sqrt{h_1^2+\cdots+h_m^2}}{2\sqrt{x_1^2+\cdots+x_m^2}}\right|+\left|\frac{x_1h_1+\cdots+x_mh_m}{2(x_1^2+\cdots+x_m^2)}\right|\to 0\,(as\ |\boldsymbol{h}|\to 0)$$

于是 $\left|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})-\varphi'(r)\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\cdot\boldsymbol{h}\right|=o(|\boldsymbol{h}|)$

于是 $f(\boldsymbol{x})$ 在 $a<|\boldsymbol{x}|<b$ 上可微, 且 $\nabla f(\boldsymbol{x})=\varphi'(r)\frac{\boldsymbol{x}}{r}$. □

9. (2) Check: $\frac{\partial f}{\partial r}(\boldsymbol{x})=\varphi'(r)$

$$\frac{\partial f}{\partial r}(\boldsymbol{x})=\nabla f(\boldsymbol{x})\cdot\boldsymbol{e}_r=\varphi'(r)\frac{\boldsymbol{x}}{r}\cdot\frac{\boldsymbol{x}}{r}\overset{\frac{\boldsymbol{x}}{r}\text{是单位向量}}{=}\varphi'(r).\ \square$$

9. (3) Check: $\frac{\partial f}{\partial \boldsymbol{l}}(\boldsymbol{x}_0)=0$, 其中 $\boldsymbol{l}\perp\boldsymbol{e}_r$.

$$\frac{\partial f}{\partial \boldsymbol{l}}(\boldsymbol{x}_0)=\nabla f(\boldsymbol{x}_0)\cdot\boldsymbol{l}=\varphi'(r)\frac{\boldsymbol{x}}{r}\cdot\boldsymbol{l}=\varphi'(r)\boldsymbol{e}_r\cdot\boldsymbol{l}=0.\ \square$$

定理 15.3.3(隐函数定理)　设 Ω 是 $\mathbf{R}^{m+1}$ 中的一个开集, $F(x,y)$ 是 Ω 上的连续函数, 它在 Ω 中的每个点都关于变元 y 可导, 并且偏导数 $F_y(x,y)$ 在 Ω 上连续. 又设 $(x_0,y_0)\in\Omega$, 且

$$F(x_0,y_0)=0,\quad F_y(x_0,y_0)\neq 0.$$

则存在 $a>0$ 和 $r>0$, 这里 a 和 r 比较小使得 $B(x_0,r)\times(y_0-a,y_0+a)\subseteq\Omega$, 和定义在 $B(x_0,r)\subseteq\mathbf{R}^m$ 上而取值于 (y_0-a,y_0+a) 的函数 $y=f(x)$, 使以下结论成立:

(1) $y=f(x)$ 通过点 (x_0,y_0) 并满足方程 $F(x,y)=0$, 即

$$f(x_0)=y_0,\quad F(x,f(x))=0,\quad \forall x\in B(x_0,r),$$

而且对每个 $x\in B(x_0,r)$, $y=f(x)$ 都是该方程在区间 (y_0-a,y_0+a) 上的唯一解;

(2) $f(x)$ 在 $B(x_0,r)$ 上连续.

进一步, 如果 $F(x,y)$ 在 Ω 中的每个点还关于每个变元 x_i $(i=1,2,\cdots,m)$ 有偏导数, 则相应地隐函数 $f(x)$ 也在 $B(x_0,r)$ 上关于每个变元 x_i $(i=1,2,\cdots,m)$ 有偏导数, 这些偏导数可按公式 (15.3.8) 计算. 而且如果每个偏导数 $F_{x_i}(x,y)(i=1,2,\cdots,m)$ 都在 Ω 上连续, 则相应地偏导数 $f_{x_i}(x)$ $(i=1,2,\cdots,m)$ 也都在 $B(x_0,r)$ 上连续.

10. 证明: 如果把隐函数定理中最后的条件 “每个偏导数 $F_{x_i}(x,y)$ $(i=1,2,\cdots,m)$ 都在 Ω 上连续” 减弱为 $F(x,y)$ 对每个固定的 y 都关于 x 可微, 则相应的隐函数 $f(x)$ 在 $B(x_0,r)$ 上可微.

10.我们考虑$F:\mathbb{R}^{m+1}\to\mathbb{R}, f:\mathbb{R}^m\to\mathbb{R}. F\in C(\Omega), f\in C(B(\boldsymbol{x}_0,r))$,$F$关于$y$连续可微,对于给定的$y$,$F$关于$\boldsymbol{x}$可微.

对于任意给定的$y\in\mathbb{R}$,对于任意$\boldsymbol{x}\in B(\boldsymbol{x}_0,r)$,由于$B(\boldsymbol{x}_0,r)$是开球,故存在$\boldsymbol{x}$的邻域$B(\boldsymbol{x},r')\subset B(\boldsymbol{x}_0,r)$.

令$\boldsymbol{h}\in B(\boldsymbol{x},r')$,于是$\boldsymbol{x}+\boldsymbol{h}\in B(\boldsymbol{x}_0,r)$,则在$\boldsymbol{h}\to\boldsymbol{0}$时,我们有

$|F(\boldsymbol{x}+\boldsymbol{h},y)-F(\boldsymbol{x},y)-(DF)(\boldsymbol{x},y)\cdot\boldsymbol{h}|=o(|\boldsymbol{h}|)$,这里$D$表示对前$m$项求偏导.

$F(\boldsymbol{x},f(\boldsymbol{x}))=0, \forall\boldsymbol{x}\in B(\boldsymbol{x}_0,r)$.

Check:$f(\boldsymbol{x})$在$B(\boldsymbol{x}_0,r)$上可微.

断言$f(\boldsymbol{x})$在$B(\boldsymbol{x}_0,r)$的微分为$((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}(DF)(\boldsymbol{x},f(\boldsymbol{x}))$.

只需要验证$|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})+((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}(DF)(\boldsymbol{x},f(\boldsymbol{x}))\cdot\boldsymbol{h}|=o(|\boldsymbol{h}|)$

我们有$|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})+((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}(DF)(\boldsymbol{x},f(\boldsymbol{x}))\cdot\boldsymbol{h}|\le|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}|\cdot|((JF)(\boldsymbol{x},f(\boldsymbol{x})))[f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})]+(DF)(\boldsymbol{x},f(\boldsymbol{x}))\cdot\boldsymbol{h}|$

J表示对第$m+1$项求偏导,于是$JF\in C(\Omega)$.在$\boldsymbol{h}\to\boldsymbol{0}$时,

$|((JF)(\boldsymbol{x},f(\boldsymbol{x})))[f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})]+(DF)(\boldsymbol{x},f(\boldsymbol{x}))\cdot\boldsymbol{h}|$

$\overset{F(\boldsymbol{x},f(\boldsymbol{x}))=0,\forall\boldsymbol{x}\in B(\boldsymbol{x}_0,r).}{=}|F(\boldsymbol{x}+\boldsymbol{h},f(\boldsymbol{x}+\boldsymbol{h}))-F(\boldsymbol{x},f(\boldsymbol{x}))+((JF)(\boldsymbol{x},f(\boldsymbol{x})))[f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})]+(DF)(\boldsymbol{x},f(\boldsymbol{x}))\cdot\boldsymbol{h}|$

$\le|F(\boldsymbol{x}+\boldsymbol{h},f(\boldsymbol{x}+\boldsymbol{h}))-F(\boldsymbol{x}+\boldsymbol{h},f(\boldsymbol{x}))+((JF)(\boldsymbol{x}+\boldsymbol{h},f(\boldsymbol{x})))[f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})]|$

$+|((JF)(\boldsymbol{x},f(\boldsymbol{x})))[f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})]-((JF)(\boldsymbol{x}+\boldsymbol{h},f(\boldsymbol{x})))[f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})]|$

$+|F(\boldsymbol{x}+\boldsymbol{h},f(\boldsymbol{x}))-F(\boldsymbol{x},f(\boldsymbol{x}))+(DF)(\boldsymbol{x},f(\boldsymbol{x}))\cdot\boldsymbol{h}|$

$\overset{\substack{F关于y连续可导\\ 对于给定的y,F关于\boldsymbol{x}可微.}}{=}o(|\boldsymbol{h}|)+|((JF)(\boldsymbol{x},f(\boldsymbol{x})))-((JF)(\boldsymbol{x}+\boldsymbol{h},f(\boldsymbol{x})))||f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})|+o(|\boldsymbol{h}|)$

$\overset{F关于y连续可导\Rightarrow JF\in C(\Omega)}{=}o(|\boldsymbol{h}|)+o(1)\cdot|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})|+o(|\boldsymbol{h}|)=o(|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})|)+o(|\boldsymbol{h}|)$

只需证$o(|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})|)=o(|\boldsymbol{h}|)$

于是对于任意给定的$0<\epsilon<\dfrac{1}{2|(JF(\boldsymbol{x},f(\boldsymbol{x})))^{-1}|}$,存在$B^m(\boldsymbol{0},\tau)=\{\boldsymbol{x}\in\mathbb{R}^m:|\boldsymbol{x}|<\tau\}$,使得对于任意$\boldsymbol{h}\in B^m(\boldsymbol{0},\tau)$,

有$|(JF(\boldsymbol{x},f(\boldsymbol{x})))[f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})]+(DF)(\boldsymbol{x},f(\boldsymbol{x}))\cdot\boldsymbol{h}|\le\epsilon|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})|+\epsilon|\boldsymbol{h}|$

另一方面,$|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})+((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}(DF)(\boldsymbol{x},f(\boldsymbol{x}))\cdot\boldsymbol{h}|$

$\ge|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})|-|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}(DF)(\boldsymbol{x},f(\boldsymbol{x}))\cdot\boldsymbol{h}|$

$\ge|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})|-|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}(DF)(\boldsymbol{x},f(\boldsymbol{x}))|\cdot|\boldsymbol{h}|$

$\Rightarrow|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})|\le|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})+((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}(DF)(\boldsymbol{x},f(\boldsymbol{x}))\cdot\boldsymbol{h}|+|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}(DF)(\boldsymbol{x},f(\boldsymbol{x}))|\cdot|\boldsymbol{h}|$

$\le\epsilon|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}||f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})|+\epsilon|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}|\cdot|\boldsymbol{h}|+|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}(DF)(\boldsymbol{x},f(\boldsymbol{x}))|\cdot|\boldsymbol{h}|$

$\Rightarrow|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})|\le\dfrac{\epsilon|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}|+|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}(DF)(\boldsymbol{x},f(\boldsymbol{x}))|}{1-\epsilon|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}|}\cdot|\boldsymbol{h}|$

于是$|((JF)(\boldsymbol{x},f(\boldsymbol{x})))[f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})]+(DF)(\boldsymbol{x},f(\boldsymbol{x}))\cdot\boldsymbol{h}|\le\epsilon|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})|+\epsilon|\boldsymbol{h}|$

$\le\epsilon\dfrac{\epsilon|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}|+|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}(DF)(\boldsymbol{x},f(\boldsymbol{x}))|}{1-\epsilon|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}|}\cdot|\boldsymbol{h}|+\epsilon|\boldsymbol{h}|$

$=\epsilon\cdot\dfrac{1+|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}(DF)(\boldsymbol{x},f(\boldsymbol{x}))|}{1-\epsilon|((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}|}\cdot|\boldsymbol{h}|$

于是$|((JF)(\boldsymbol{x},f(\boldsymbol{x})))[f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})]+(DF)(\boldsymbol{x},f(\boldsymbol{x}))\cdot\boldsymbol{h}|=o(|\boldsymbol{h}|)$

故$|f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})+((JF)(\boldsymbol{x},f(\boldsymbol{x})))^{-1}(DF)(\boldsymbol{x},f(\boldsymbol{x}))\cdot\boldsymbol{h}|=o(|\boldsymbol{h}|)$

因此,$f(\boldsymbol{x})$在$B(\boldsymbol{x}_0,r)$上可微.$\square$

$$11.(1)\ F(x-au,y-bu)=0\Rightarrow\begin{cases}\dfrac{\partial F(x-au,y-bu)}{\partial x}=0\\[2mm]\dfrac{\partial F(x-au,y-bu)}{\partial y}=0\end{cases}$$

$$\Rightarrow\begin{cases}\dfrac{\partial F(x-au,y-bu)}{\partial(x-au)}\dfrac{\partial(x-au)}{\partial x}+\dfrac{\partial F(x-au,y-bu)}{\partial(y-bu)}\dfrac{\partial(y-bu)}{\partial x}=0\\[2mm]\dfrac{\partial F(x-au,y-bu)}{\partial(x-au)}\dfrac{\partial(x-au)}{\partial y}+\dfrac{\partial F(x-au,y-bu)}{\partial(y-bu)}\dfrac{\partial(y-bu)}{\partial y}=0\end{cases}$$

$$\Rightarrow\begin{cases}D_1F(x-au,y-bu)\left(1-a\dfrac{\partial u}{\partial x}\right)-D_2F(x-au,y-bu)\cdot b\dfrac{\partial u}{\partial x}=0\\[2mm]-D_1F(x-au,y-bu)\cdot a\dfrac{\partial u}{\partial y}+D_2F(x-au,y-bu)\left(1-b\dfrac{\partial u}{\partial y}\right)=0\end{cases}$$

$$\Rightarrow\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{D_1F(x-au,y-bu)}{aD_1F(x-au,y-bu)+bD_2F(x-au,y-bu)}\\[2mm]\dfrac{\partial u}{\partial y}=\dfrac{D_2F(x-au,y-bu)}{aD_1F(x-au,y-bu)+bD_2F(x-au,y-bu)}\end{cases}$$

$$\Rightarrow a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}=\frac{aD_1F(x-au,y-bu)}{aD_1F(x-au,y-bu)+bD_2F(x-au,y-bu)}+\frac{bD_2F(x-au,y-bu)}{aD_1F(x-au,y-bu)+bD_2F(x-au,y-bu)}=1.\square$$

$11.(4)\ x^2+y^2+u^2=yf\left(\frac{u}{y}\right)$,记$H(x,y)=x^2+y^2+u^2-yf\left(\frac{u}{y}\right)=0$

$$\Rightarrow\begin{cases}\dfrac{\partial H(x,y)}{\partial x}=0\\[2mm]\dfrac{\partial H(x,y)}{\partial y}=0\end{cases}\Rightarrow\begin{cases}2x+2u\dfrac{\partial u}{\partial x}-y\dfrac{\partial f\left(\frac{u}{y}\right)}{\partial\frac{u}{y}}\dfrac{\partial\frac{u}{y}}{\partial x}=0\\[2mm]2y+2u\dfrac{\partial u}{\partial y}-f\left(\dfrac{u}{y}\right)-y\dfrac{\partial f\left(\frac{u}{y}\right)}{\partial\frac{u}{y}}\dfrac{\partial\frac{u}{y}}{\partial y}=0\end{cases}$$

$$\Rightarrow\begin{cases}2x+2u\dfrac{\partial u}{\partial x}-f'\left(\dfrac{u}{y}\right)\dfrac{\partial u}{\partial x}=0\\[2mm]2y+2u\dfrac{\partial u}{\partial y}-f\left(\dfrac{u}{y}\right)-yf'\left(\dfrac{u}{y}\right)\left(\dfrac{\partial u}{\partial y}\dfrac{1}{y}-\dfrac{u}{y^2}\right)=0\end{cases}$$

$$\Rightarrow\begin{cases}2x+2u\dfrac{\partial u}{\partial x}-f'\left(\dfrac{u}{y}\right)\dfrac{\partial u}{\partial x}=0\\[2mm]2y+2u\dfrac{\partial u}{\partial y}-f\left(\dfrac{u}{y}\right)-f'\left(\dfrac{u}{y}\right)\left(\dfrac{\partial u}{\partial y}-\dfrac{u}{y}\right)=0\end{cases}$$

$$\Rightarrow\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{2x}{2u-f'\left(\frac{u}{y}\right)}\\[2mm]\dfrac{\partial u}{\partial y}=\dfrac{f\left(\frac{u}{y}\right)-\frac{u}{y}f'\left(\frac{u}{y}\right)-2y}{2u-f'\left(\frac{u}{y}\right)}\end{cases}$$

$$\Rightarrow(x^2-y^2-u^2)\frac{\partial u}{\partial x}+2xy\frac{\partial u}{\partial y}-2xu=(x^2-y^2-u^2)\frac{-2x}{2u-f'\left(\frac{u}{y}\right)}+2xy\frac{f\left(\frac{u}{y}\right)-\frac{u}{y}f'\left(\frac{u}{y}\right)-2y}{2u-f'\left(\frac{u}{y}\right)}-2xu$$

$$=\frac{-2x^3+2xy^2+2xu^2}{2u-f'\left(\frac{u}{y}\right)}+\frac{2xyf\left(\frac{u}{y}\right)-2uxf'\left(\frac{u}{y}\right)-4xy^2}{2u-f'\left(\frac{u}{y}\right)}-2xu$$

$$=\frac{-2x^3+2xy^2+2xu^2+2xyf\left(\frac{u}{y}\right)-2uxf'\left(\frac{u}{y}\right)-4xy^2-4xu^2+2xuf'\left(\frac{u}{y}\right)}{2u-f'\left(\frac{u}{y}\right)}$$

$$=\frac{-2x^3-2xy^2-2xu^2+2xyf\left(\frac{u}{y}\right)}{2u-f'\left(\frac{u}{y}\right)}=0.\square$$

12. $F(x,y,z)=0$

$$x=x(y,z)\Rightarrow F(x(y,z),y,z)=0\Rightarrow\begin{cases}\dfrac{\partial F(x(y,z),y,z)}{\partial y}=0\\\dfrac{\partial F(x(y,z),y,z)}{\partial z}=0\end{cases}\Rightarrow\begin{cases}F_1(x,y,z)\dfrac{\partial x}{\partial y}+F_2(x,y,z)=0\\F_1(x,y,z)\dfrac{\partial x}{\partial z}+F_3(x,y,z)=0\end{cases}$$

$$y=y(x,z)\Rightarrow F(x,y(x,z),z)=0\Rightarrow\begin{cases}\dfrac{\partial F(x,y(x,z),z)}{\partial x}=0\\\dfrac{\partial F(x,y(x,z),z)}{\partial z}=0\end{cases}\Rightarrow\begin{cases}F_1(x,y,z)+F_2(x,y,z)\dfrac{\partial y}{\partial x}=0\\F_2(x,y,z)\dfrac{\partial y}{\partial z}+F_3(x,y,z)=0\end{cases}$$

$$z=z(x,y)\Rightarrow F(x,y,z(x,y))=0\Rightarrow\begin{cases}\dfrac{\partial F(x,y,z(x,y))}{\partial x}=0\\\dfrac{\partial F(x,y,z(x,y))}{\partial y}=0\end{cases}\Rightarrow\begin{cases}F_1(x,y,z)+F_3(x,y,z)\dfrac{\partial z}{\partial x}=0\\F_2(x,y,z)+F_3(x,y,z)\dfrac{\partial z}{\partial y}=0\end{cases}$$

$$\Rightarrow\begin{cases}\dfrac{\partial x}{\partial y}=-\dfrac{F_2(x,y,z)}{F_1(x,y,z)}\\\dfrac{\partial x}{\partial z}=-\dfrac{F_3(x,y,z)}{F_1(x,y,z)}\\\dfrac{\partial y}{\partial x}=-\dfrac{F_1(x,y,z)}{F_2(x,y,z)}\\\dfrac{\partial y}{\partial z}=-\dfrac{F_3(x,y,z)}{F_2(x,y,z)}\\\dfrac{\partial z}{\partial x}=-\dfrac{F_1(x,y,z)}{F_3(x,y,z)}\\\dfrac{\partial z}{\partial y}=-\dfrac{F_2(x,y,z)}{F_3(x,y,z)}\end{cases}\Rightarrow\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial y}=-1,\ \frac{\partial x}{\partial z}\cdot\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial x}=-1.\square$$

$$1.(3)\ \frac{\partial^2 f(x,y)}{\partial x^2}=\frac{\partial^2 \ln(x^2+y^2)}{\partial x^2}=\frac{\partial \frac{2x}{x^2+y^2}}{\partial x}=\frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f(x,y)}{\partial y^2}=\frac{\partial^2 \ln(x^2+y^2)}{\partial y^2}=\frac{\partial \frac{2y}{x^2+y^2}}{\partial y}=\frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f(x,y)}{\partial x\partial y}=\frac{\partial \frac{2y}{x^2+y^2}}{\partial x}=-\frac{4xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f(x,y)}{\partial y\partial x}=\frac{\partial \frac{2x}{x^2+y^2}}{\partial y}=-\frac{4xy}{(x^2+y^2)^2}$$

$$1.(5)\ \frac{\partial^2 f(x,y)}{\partial x^2}=\frac{\partial^2 e^{\arctan\frac{y}{x}}}{\partial x^2}=\frac{\partial e^{\arctan\frac{y}{x}}\left(\frac{1}{1+\left(\frac{y}{x}\right)^2}\right)\frac{-y}{x^2}}{\partial x}=\frac{\partial e^{\arctan\frac{y}{x}}\frac{-y}{x^2+y^2}}{\partial x}$$

$$=e^{\arctan\frac{y}{x}}\frac{-y}{x^2+y^2}\frac{-y}{x^2+y^2}+e^{\arctan\frac{y}{x}}\frac{2xy}{(x^2+y^2)^2}=e^{\arctan\frac{y}{x}}\frac{y^2}{(x^2+y^2)^2}+e^{\arctan\frac{y}{x}}\frac{2xy}{(x^2+y^2)^2}$$

$$=\frac{y^2+2xy}{(x^2+y^2)^2}e^{\arctan\frac{y}{x}}$$

$$\frac{\partial^2 f(x,y)}{\partial y^2}=\frac{\partial^2 e^{\arctan\frac{y}{x}}}{\partial y^2}=\frac{\partial e^{\arctan\frac{y}{x}}\left(\frac{1}{1+\left(\frac{y}{x}\right)^2}\right)\frac{1}{x}}{\partial x}=\frac{\partial e^{\arctan\frac{y}{x}}\frac{x}{x^2+y^2}}{\partial x}$$

$$=e^{\arctan\frac{y}{x}}\left(\frac{x}{x^2+y^2}\right)^2+e^{\arctan\frac{y}{x}}\frac{-2xy}{(x^2+y^2)^2}=e^{\arctan\frac{y}{x}}\frac{x^2-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f(x,y)}{\partial x\partial y}=\frac{\partial e^{\arctan\frac{y}{x}}\frac{x}{x^2+y^2}}{\partial x}=e^{\arctan\frac{y}{x}}\frac{-y}{x^2+y^2}\frac{x}{x^2+y^2}+e^{\arctan\frac{y}{x}}\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}$$

$$=e^{\arctan\frac{y}{x}}\frac{y^2-xy-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f(x,y)}{\partial y\partial x}=\frac{\partial e^{\arctan\frac{y}{x}}\frac{-y}{x^2+y^2}}{\partial y}=e^{\arctan\frac{y}{x}}\frac{x}{x^2+y^2}\frac{-y}{x^2+y^2}+e^{\arctan\frac{y}{x}}\frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2}$$

$$=e^{\arctan\frac{y}{x}}\frac{y^2-xy-x^2}{(x^2+y^2)^2}$$

$2.(2) f(x,y)=x^3\sin^2 y$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}=\frac{\partial^2 2x^3(\cos^2 y-\sin^2 y)}{\partial x^2}=12x\cos 2y$$

$$\frac{\partial^4 f}{\partial x \partial y^2 \partial x}=\frac{\partial^4 x^3\sin^2 y}{\partial x \partial y^2 \partial x}=\frac{\partial^3 3x^2\sin^2 y}{\partial x \partial y^2}=\frac{\partial 6x^2(\cos^2 y-\sin^2 y)}{\partial x}=12x\cos 2y$$

$$\frac{\partial^4 f}{\partial y^2 \partial x^2}=\frac{\partial^2 6x\sin^2 y}{\partial y^2}=12x\cos 2y$$

$2.(3) f(x,y,z)=e^{xyz}$

$$\frac{\partial^3 f}{\partial x \partial y \partial z}=\frac{\partial^3 e^{xyz}}{\partial x \partial y \partial z}=\frac{\partial^2 xye^{xyz}}{\partial x \partial y}=\frac{\partial(xe^{xyz}+x^2yze^{xyz})}{\partial x}$$

$$=(1+xyz)e^{xyz}+2xyze^{xyz}+x^2y^2z^2e^{xyz}=(1+3xyz+x^2y^2z^2)e^{xyz}$$

$$\frac{\partial^3 f}{\partial z \partial x \partial y}=\frac{\partial^3 e^{xyz}}{\partial z \partial x \partial y}=\frac{\partial^2 xze^{xyz}}{\partial z \partial x}=\frac{\partial(ze^{xyz}+xyz^2e^{xyz})}{\partial x}=(1+3xyz+x^2y^2z^2)e^{xyz}$$

$$\frac{\partial^3 f}{\partial y \partial z \partial x}=\frac{\partial^3 e^{xyz}}{\partial y \partial z \partial x}=\frac{\partial^2 yze^{xyz}}{\partial y \partial z}=\frac{\partial(ye^{xyz}+xy^2ze^{xyz})}{\partial y}=(1+3xyz+x^2y^2z^2)e^{xyz}$$

4.(1) $\lim\limits_{\substack{(x,y)\to(0,0)\\x=ky}} f(x,y)=\lim\limits_{\substack{(x,y)\to(0,0)\\x=ky}}\frac{xy}{x^2+y^2}=\lim\limits_{\substack{(x,y)\to(0,0)\\x=ky}}\frac{kx^2}{x^2+k^2x^2}=\frac{k}{1+k^2}\neq 0(k\neq 0)$

故f在$(0,0)$不连续.

$$\frac{\partial^n f}{\partial x^n}(0,0)=\lim_{\Delta x\to 0}\frac{\frac{\partial^{n-1}f}{\partial x^{n-1}}(\Delta x,0)-\frac{\partial^{n-1}f}{\partial x^{n-1}}(0,0)}{\Delta x}$$

$$\frac{\partial^n f}{\partial x^n}=\frac{\partial^n\frac{xy}{x^2+y^2}}{\partial x^n}=y\frac{\partial^n x\frac{1}{x^2+y^2}}{\partial x^n}=y\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^k x}{\partial x^k}\frac{\partial^{n-k}\frac{1}{x^2+y^2}}{\partial x^{n-k}}$$

$$=y\left[x\frac{\partial^n\frac{1}{x^2+y^2}}{\partial x^n}+n\frac{\partial^{n-1}\frac{1}{x^2+y^2}}{\partial x^{n-1}}\right]$$

考虑复数上裂项: $\frac{\partial^n\frac{1}{x^2+y^2}}{\partial x^n}=\frac{1}{2iy}\frac{\partial^n\left(\frac{1}{x-iy}-\frac{1}{x+iy}\right)}{\partial x^n}=\frac{1}{2iy}\left[\frac{\partial^n\left(\frac{1}{x-iy}\right)}{\partial x^n}-\frac{\partial^n\left(\frac{1}{x+iy}\right)}{\partial x^n}\right]$

$$=\frac{(-1)^n(n!)}{2iy}\left[\frac{1}{(x-iy)^{n+1}}-\frac{1}{(x+iy)^{n+1}}\right]=\frac{(-1)^n(n!)}{2iy(x^2+y^2)^{n+1}}[(x+iy)^{n+1}-(x-iy)^{n+1}]$$

$$=\frac{(-1)^n(n!)}{2iy(x^2+y^2)^{\frac{n+1}{2}}}\left(e^{i(n+1)\arctan\frac{y}{x}}-e^{-i(n+1)\arctan\frac{y}{x}}\right)=\frac{(-1)^n(n!)}{y(x^2+y^2)^{\frac{n+1}{2}}}\sin\left((n+1)\arctan\frac{y}{x}\right)$$

于是 $\frac{\partial^n f}{\partial x^n}=x\frac{(-1)^n(n!)}{(x^2+y^2)^{\frac{n+1}{2}}}\sin\left((n+1)\arctan\frac{y}{x}\right)+n\frac{(-1)^{n-1}(n-1)!}{(x^2+y^2)^{\frac{n}{2}}}\sin\left(n\arctan\frac{y}{x}\right)$

类似地 $\frac{\partial^n f}{\partial y^n}=y\frac{(-1)^n(n!)}{(x^2+y^2)^{\frac{n+1}{2}}}\sin\left((n+1)\arctan\frac{x}{y}\right)+n\frac{(-1)^{n-1}(n-1)!}{(x^2+y^2)^{\frac{n}{2}}}\sin\left(n\arctan\frac{x}{y}\right)$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$$

$$\frac{\partial^{n+1} f}{\partial x^{n+1}}(0,0)=\lim_{\Delta x\to 0}\frac{\frac{\partial^n f}{\partial x^n}(\Delta x,0)-\frac{\partial^n f}{\partial x^n}(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0$$

$$\frac{\partial^{n+1} f}{\partial y^{n+1}}(0,0)=\lim_{\Delta y\to 0}\frac{\frac{\partial^n f}{\partial y^n}(0,\Delta y)-\frac{\partial^n f}{\partial y^n}(0,0)}{\Delta y}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta y}=0$$

4.(2) $f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^2},(x,y)\neq(0,0)\\ 0,(x,y)=(0,0)\end{cases}$

$$\frac{\partial f}{\partial y}=\frac{\partial \frac{xy^2}{x^2+y^2}}{\partial y}=\frac{2xy}{x^2+y^2}-\frac{2xy^3}{(x^2+y^2)^2}=\frac{2x^3y}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0-0}{\Delta y}=0$$

$$\frac{\partial f}{\partial x}=\frac{\partial \frac{xy^2}{x^2+y^2}}{\partial x}=\frac{y^2}{x^2+y^2}-\frac{2x^2y^2}{(x^2+y^2)^2}=\frac{y^4-x^2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0$$

$$\frac{\partial^2 f}{\partial x\partial y}(0,0)=\lim_{\Delta x\to 0}\frac{\frac{\partial f}{\partial y}(\Delta x,0)-\frac{\partial f}{\partial y}(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0$$

$$\frac{\partial^2 f}{\partial y\partial x}(0,0)=\lim_{\Delta y\to 0}\frac{\frac{\partial f}{\partial x}(0,\Delta y)-\frac{\partial f}{\partial x}(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{\frac{\partial f}{\partial x}(0,\Delta y)-\frac{\partial f}{\partial x}(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{1}{\Delta y}=\infty \text{故不存在.}$$

(3) $f(x,y)=\begin{cases}y^3\ln(x^2+y^2),(x,y)\neq(0,0)\\ 0,(x,y)=(0,0)\end{cases}$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{(\Delta y)^3\ln((\Delta y)^2)}{\Delta y}=0$$

$$\frac{\partial f}{\partial y}=\frac{\partial(y^3\ln(x^2+y^2))}{\partial y}=3y^2\ln(x^2+y^2)+\frac{2y^4}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x\partial y}(0,0)=\lim_{\Delta x\to 0}\frac{\frac{\partial f}{\partial y}(\Delta x,0)-\frac{\partial f}{\partial y}(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial\left[3y^2\ln(x^2+y^2)+\frac{2y^4}{x^2+y^2}\right]}{\partial x}=3y^2\frac{2x}{x^2+y^2}-\frac{4xy^4}{(x^2+y^2)^2}=\frac{6x^3y^2+2xy^4}{(x^2+y^2)^2}$$

$$\frac{\partial^3 f}{\partial^2 x\partial y}(0,0)=\lim_{\Delta x\to 0}\frac{\frac{\partial^2 f}{\partial x\partial y}(\Delta x,0)-\frac{\partial^2 f}{\partial x\partial y}(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$$

$$\frac{\partial f}{\partial x}=\frac{\partial(y^3\ln(x^2+y^2))}{\partial x}=\frac{2xy^3}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2}(0,0)=\lim_{\Delta x\to 0}\frac{\frac{\partial f}{\partial x}(\Delta x,0)-\frac{\partial f}{\partial x}(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial\frac{2xy^3}{x^2+y^2}}{\partial x}=\frac{2y^3}{x^2+y^2}-\frac{4x^2y^3}{(x^2+y^2)^2}=\frac{2y^3(x^2+y^2)-4x^2y^3}{(x^2+y^2)^2}=\frac{2y^5-2x^2y^3}{(x^2+y^2)^2}$$

$$\frac{\partial^3 f}{\partial y\partial^2 x}(0,0)=\lim_{\Delta y\to 0}\frac{\frac{\partial^2 f}{\partial x^2}(0,\Delta y)-\frac{\partial^2 f}{\partial x^2}(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{\frac{2(\Delta y)^5}{(\Delta y)^4}}{\Delta y}=2\neq\frac{\partial^3 f}{\partial^2 x\partial y}(0,0)$$

$12.(2)\, u = x\varphi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$

验证 $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial\left[x\varphi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)\right]}{\partial x} = \varphi\left(\frac{y}{x}\right) + x\varphi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + \psi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$= \varphi\left(\frac{y}{x}\right) - \frac{y}{x}\varphi'\left(\frac{y}{x}\right) - \frac{y}{x^2}\psi'\left(\frac{y}{x}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial\left[\varphi\left(\frac{y}{x}\right) - \frac{y}{x}\varphi'\left(\frac{y}{x}\right) - \frac{y}{x^2}\psi'\left(\frac{y}{x}\right)\right]}{\partial x}$$

$$= \varphi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + \frac{y}{x^2}\varphi'\left(\frac{y}{x}\right) - \frac{y}{x}\varphi''\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + \frac{2y}{x^3}\psi'\left(\frac{y}{x}\right) - \frac{y}{x^2}\psi''\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$= \frac{y^2}{x^3}\varphi''\left(\frac{y}{x}\right) + \frac{2y}{x^3}\psi'\left(\frac{y}{x}\right) + \frac{y^2}{x^4}\psi''\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial\left[x\varphi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)\right]}{\partial y} = \varphi'\left(\frac{y}{x}\right) + \frac{1}{x}\psi'\left(\frac{y}{x}\right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial\left[\varphi'\left(\frac{y}{x}\right) + \frac{1}{x}\psi'\left(\frac{y}{x}\right)\right]}{\partial x} = -\frac{y}{x^2}\varphi''\left(\frac{y}{x}\right) - \frac{1}{x^2}\psi'\left(\frac{y}{x}\right) - \frac{y}{x^3}\psi''\left(\frac{y}{x}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial\left[\varphi'\left(\frac{y}{x}\right) + \frac{1}{x}\psi'\left(\frac{y}{x}\right)\right]}{\partial y} = \frac{1}{x}\varphi''\left(\frac{y}{x}\right) + \frac{1}{x^2}\psi''\left(\frac{y}{x}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

$$= x^2\left[\frac{y^2}{x^3}\varphi''\left(\frac{y}{x}\right) + \frac{2y}{x^3}\psi'\left(\frac{y}{x}\right) + \frac{y^2}{x^4}\psi''\left(\frac{y}{x}\right)\right] + 2xy\left[-\frac{y}{x^2}\varphi''\left(\frac{y}{x}\right) - \frac{1}{x^2}\psi'\left(\frac{y}{x}\right) - \frac{y}{x^3}\psi''\left(\frac{y}{x}\right)\right] + y^2\left[\frac{1}{x}\varphi''\left(\frac{y}{x}\right) + \frac{1}{x^2}\psi''\left(\frac{y}{x}\right)\right]$$

$$= \frac{y^2}{x}\varphi''\left(\frac{y}{x}\right) + \frac{2y}{x}\psi'\left(\frac{y}{x}\right) + \frac{y^2}{x^2}\psi''\left(\frac{y}{x}\right) - \frac{2y^2}{x}\varphi''\left(\frac{y}{x}\right) - \frac{2y}{x}\psi'\left(\frac{y}{x}\right) - \frac{2y^2}{x^2}\psi''\left(\frac{y}{x}\right) + \frac{y^2}{x}\varphi''\left(\frac{y}{x}\right) + \frac{y^2}{x^2}\psi''\left(\frac{y}{x}\right)$$

$= 0.\square$

$12.(4)\, x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$，其中 $u = \varphi(xy) + \psi\left(\frac{x}{y}\right)$

$$\frac{\partial u}{\partial x} = \frac{\partial\left[\varphi(xy) + \psi\left(\frac{x}{y}\right)\right]}{\partial x} = y\varphi'(xy) + \frac{1}{y}\psi'\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial\left[y\varphi'(xy) + \frac{1}{y}\psi'\left(\frac{x}{y}\right)\right]}{\partial x} = y^2\varphi''(xy) + \frac{1}{y^2}\psi''\left(\frac{x}{y}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial\left[\varphi(xy) + \psi\left(\frac{x}{y}\right)\right]}{\partial y} = x\varphi'(xy) - \frac{x}{y^2}\psi'\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial\left[x\varphi'(xy) - \frac{x}{y^2}\psi'\left(\frac{x}{y}\right)\right]}{\partial y} = x^2\varphi''(xy) + \frac{2x}{y^3}\psi'\left(\frac{x}{y}\right) + \frac{x^2}{y^4}\psi''\left(\frac{x}{y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}$$

$$= x^2\left[y^2\varphi''(xy) + \frac{1}{y^2}\psi''\left(\frac{x}{y}\right)\right] - y^2\left[x^2\varphi''(xy) + \frac{2x}{y^3}\psi'\left(\frac{x}{y}\right) + \frac{x^2}{y^4}\psi''\left(\frac{x}{y}\right)\right] + x\left[y\varphi'(xy) + \frac{1}{y}\psi'\left(\frac{x}{y}\right)\right] - y\left[x\varphi'(xy) - \frac{x}{y^2}\psi'\left(\frac{x}{y}\right)\right]$$

$= 0.\square$

1.(1) $z=ax^2+by^2(a,b>0)$

切平面: $z-z_0=2ax_0(x-x_0)+2by_0(y-y_0)$

单位法向量: $\vec{\boldsymbol{n}}=\pm\dfrac{(2ax_0,2by_0,-1)}{\sqrt{4a^2x_0^2+4b^2y_0^2+1}}$

1.(2) $z=ax^2-by^2(a,b>0)$

切平面: $z-z_0=2ax_0(x-x_0)-2by_0(y-y_0)$

单位法向量: $\vec{\boldsymbol{n}}=\pm\dfrac{(2ax_0,-2by_0,-1)}{\sqrt{4a^2x_0^2+4b^2y_0^2+1}}$

1.(3) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1(a,b,c>0)$

切平面: $\dfrac{2x_0}{a^2}(x-x_0)+\dfrac{2y_0}{b^2}(y-y_0)+\dfrac{2z_0}{c^2}(z-z_0)=0$

单位法向量: $\vec{\boldsymbol{n}}=\pm\dfrac{\left(\frac{2x_0}{a^2},\frac{2y_0}{b^2},\frac{2z_0}{c^2}\right)}{\sqrt{\frac{4x_0^2}{a^4}+\frac{4y_0^2}{b^4}+\frac{4z_0^2}{c^4}}}$

1.(4) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1(a,b,c>0)$

切平面: $\dfrac{2x_0}{a^2}(x-x_0)+\dfrac{2y_0}{b^2}(y-y_0)-\dfrac{2z_0}{c^2}(z-z_0)=0$

单位法向量: $\vec{\boldsymbol{n}}=\pm\dfrac{\left(\frac{2x_0}{a^2},\frac{2y_0}{b^2},-\frac{2z_0}{c^2}\right)}{\sqrt{\frac{4x_0^2}{a^4}+\frac{4y_0^2}{b^4}+\frac{4z_0^2}{c^4}}}$

1.(5) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a,b>0)$

切平面: $\dfrac{2x_0}{a^2}(x-x_0)+\dfrac{2y_0}{b^2}(y-y_0)=0$

单位法向量: $\vec{\boldsymbol{n}}=\pm\dfrac{\left(\frac{2x_0}{a^2},\frac{2y_0}{b^2},0\right)}{\sqrt{\frac{4x_0^2}{a^4}+\frac{4y_0^2}{b^4}}}$

4. $xyz=a^3(a>0)$

切平面: $y_0z_0(x-x_0)+x_0z_0(y-y_0)+x_0y_0(z-z_0)=0$

$\Rightarrow y_0z_0x+x_0z_0y+x_0y_0z=3x_0y_0z_0=3a^3$

该切平面与坐标轴交于: $\left(\dfrac{3a^3}{y_0z_0},0,0\right),\left(0,\dfrac{3a^3}{x_0z_0},0\right),\left(0,0,\dfrac{3a^3}{x_0y_0}\right)$

$V=\dfrac{1}{6}\left|\dfrac{3a^3}{y_0z_0}\dfrac{3a^3}{x_0z_0}\dfrac{3a^3}{x_0y_0}\right|=\dfrac{9a^3}{2}$

于是曲面 $xyz=a^3(a>0)$ 的切平面与坐标面形成体积一定的四面体

5. $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}(a>0)$

切平面: $\dfrac{1}{2\sqrt{x_0}}(x-x_0)+\dfrac{1}{2\sqrt{y_0}}(y-y_0)+\dfrac{1}{2\sqrt{z_0}}(z-z_0)=0$

即 $\dfrac{1}{\sqrt{x_0}}x+\dfrac{1}{\sqrt{y_0}}y+\dfrac{1}{\sqrt{z_0}}z=\sqrt{a}$

该切平面与坐标轴交于: $(\sqrt{ax_0},0,0),(0,\sqrt{ay_0},0),(0,0,\sqrt{az_0})$

线段长度之和 $=\sqrt{ax_0}+\sqrt{ay_0}+\sqrt{az_0}=a.$

9.(2) $z=x^2+y^2, y^2+z^2=1$

对于 $(x_0,y_0,z_0)\in\{(x,y,z):z=x^2+y^2\}$

有 (x_0,y_0,z_0) 处切平面为 $z=z_0+2x_0(x-x_0)+2y_0(y-y_0)=2x_0x+2y_0y-z_0$

对于 $(x_0,y_0,z_0)\in\{(x,y,z):y^2+z^2=1\}$

有 (x_0,y_0,z_0) 处切平面为 $2y_0(y-y_0)+2z_0(z-z_0)=0$，即 $y_0y+z_0z=1$

于是切线方程为 $\begin{cases} z=2x_0x+2y_0y-z_0 \\ y_0y+z_0z=1 \end{cases}$

即 $\dfrac{x-x_0}{2y_0z_0+y_0}=\dfrac{y-y_0}{-2x_0z_0}=\dfrac{z-z_0}{2x_0y_0}$

9.(4) $x^2+y^2+z^2=4, z=\dfrac{1}{2}(x^2-y^2)$

对于 $(x_0,y_0,z_0)\in\{(x,y,z):x^2+y^2+z^2=4\}$

有 (x_0,y_0,z_0) 处切平面为 $2x_0(x-x_0)+2y_0(y-y_0)+2z_0(z-z_0)=0$，即 $x_0x+y_0y+z_0z=4$

对于 $(x_0,y_0,z_0)\in\left\{(x,y,z):z=\dfrac{1}{2}(x^2-y^2)\right\}$

有 (x_0,y_0,z_0) 处切平面为 $z=z_0+x_0(x-x_0)-y_0(y-y_0)$，即 $z=x_0x-y_0y-z_0$

于是切线方程为 $\begin{cases} x_0x+y_0y+z_0z=4 \\ z=x_0x-y_0y-z_0 \end{cases}$

即 $\dfrac{x-x_0}{y_0z_0-y_0}=\dfrac{y-y_0}{x_0z_0+x_0}=\dfrac{z-z_0}{-2x_0y_0}$.