$1.(1)f(x,y)=(x+y)\sin(x+y)$

$$\frac{\partial f}{\partial x}(x,y)=\sin(x+y)+(x+y)\cos(x+y)$$

$$\frac{\partial f}{\partial y}(x,y)=\sin(x+y)+(x+y)\cos(x+y)$$

$1.(5)f(x,y)=\arctan\frac{x+y}{1-xy}=\arctan x+\arctan y+k\pi,k\in\mathbb{Z}$

$$\frac{\partial f}{\partial x}(x,y)=\frac{1}{1+x^2}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{1}{1+y^2}$$

$1.(7)f(x,y,z)=\left(\frac{x}{y}\right)^z=x^zy^{-z}$

$$\frac{\partial f}{\partial x}(x,y,z)=\frac{\partial}{\partial x}(x^zy^{-z})=zx^{z-1}y^{-z}$$

$$\frac{\partial f}{\partial y}(x,y,z)=\frac{\partial}{\partial y}(x^zy^{-z})=-zx^zy^{-z-1}$$

$$\frac{\partial f}{\partial z}(x,y,z)=\frac{\partial}{\partial z}\left(\frac{x}{y}\right)^z=\left(\frac{x}{y}\right)^z\cdot\ln\left(\frac{x}{y}\right)$$

$$2.(2)f(x,y)=\begin{cases}x^2y^2\arctan\frac{1}{x}\arctan\frac{1}{y},x\neq0,y\neq0\\0,xy=0\end{cases}$$

①$x\neq0,y\neq0$时,

$$\frac{\partial f}{\partial x}(x,y)=y^2\arctan\frac{1}{y}\frac{\partial x^2\arctan\frac{1}{x}}{\partial x}=y^2\arctan\frac{1}{y}\left(2x\arctan\frac{1}{x}+x^2\cdot\frac{-\frac{1}{x^2}}{1+\frac{1}{x^2}}\right)$$

$$=y^2\arctan\frac{1}{y}\left(2x\arctan\frac{1}{x}-\frac{x^2}{1+x^2}\right)$$

$$\frac{\partial f}{\partial y}(x,y)=x^2\arctan\frac{1}{x}\left(2y\arctan\frac{1}{y}-\frac{y^2}{1+y^2}\right)$$

②$x=0,y\neq0$时,

$$\frac{\partial f}{\partial x}(x,y)=\lim_{\Delta x\to0}\frac{(\Delta x)^2y^2\arctan\frac{1}{\Delta x}\arctan\frac{1}{y}-0}{\Delta x}=\lim_{\Delta x\to0}(\Delta x)y^2\arctan\frac{1}{\Delta x}\arctan\frac{1}{y}=0$$

$$\frac{\partial f}{\partial y}(x,y)=\lim_{\Delta y\to0}\frac{0-0}{\Delta y}=0$$

③$x\neq0,y=0$时,

$$\frac{\partial f}{\partial x}(x,y)=\lim_{\Delta x\to0}\frac{0-0}{\Delta x}=0$$

$$\frac{\partial f}{\partial y}(x,y)=\lim_{\Delta y\to0}\frac{(\Delta y)^2x^2\arctan\frac{1}{x}\arctan\frac{1}{\Delta y}-0}{\Delta x}=\lim_{\Delta y\to0}(\Delta y)x^2\arctan\frac{1}{x}\arctan\frac{1}{\Delta y}=0$$

④$x=y=0$时,

$$\frac{\partial f}{\partial x}(x,y)=\lim_{\Delta x\to0}\frac{0-0}{\Delta x}=0$$

$$\frac{\partial f}{\partial y}(x,y)=\lim_{\Delta y\to0}\frac{0-0}{\Delta y}=0$$

$$2.(3)f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2},(x,y)\neq(0,0)\\ 0,(x,y)=(0,0)\end{cases}$$

①$(x,y)\neq(0,0)$时,

$$\frac{\partial f}{\partial x}(x,y)=\frac{\partial}{\partial x}\left(\frac{x^2y}{x^2+y^2}\right)=\frac{2xy^3}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{\partial}{\partial y}\left(\frac{x^2y}{x^2+y^2}\right)=\frac{x^2(x^2-y^2)}{(x^2+y^2)^2}$$

②$(x,y)=(0,0)$时,

$$\frac{\partial f}{\partial x}(x,y)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0}{\Delta x}=0$$

$$\frac{\partial f}{\partial y}(x,y)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta x\to 0}\frac{0}{\Delta y}=0$$

$$3.f(x,y)=\begin{cases}\dfrac{xy}{(x^2+y^2)^2},(x,y)\neq(0,0)\\ 0,(x,y)=(0,0)\end{cases}$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0}{\Delta x}=0$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta x\to 0}\frac{0}{\Delta y}=0$$

对于方向$\boldsymbol{l}=(\cos\theta,\sin\theta)$

$$\lim_{t\to 0^+}f(t\cos\theta,t\sin\theta)=\lim_{t\to 0^+}\frac{t^2\sin\theta\cos\theta}{t^4}=\lim_{t\to 0^+}\frac{\sin\theta\cos\theta}{t^2}\to+\infty$$

故$f$在$(0,0)$不连续.

因为$(0,0)$的任何邻域中都存在$\{(t\cos\theta,t\sin\theta):t\in\mathbb{R}^+,\theta\in[0,2\pi)\}$中的点

所以$f$在$(0,0)$的任何邻域中都无界.□

$$4.f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2},(x,y)\neq(0,0)\\ 0,(x,y)=(0,0)\end{cases}$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0}{\Delta x}=0$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta x\to 0}\frac{0}{\Delta y}=0$$

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{x^2y}{(x^2+y^2)^{\frac{3}{2}}}\text{不存在.}$$

故$f$在$(0,0)$不可微.□

8.显然$|\varphi(0)|\leq C|0|^{\mu}=0\Rightarrow\varphi(0)=0$

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{\varphi(xy)-\varphi(0)}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{\varphi(xy)}{\sqrt{x^2+y^2}}$$

$$\lim_{(x,y)\to(0,0)}\left|\frac{\varphi(xy)}{\sqrt{x^2+y^2}}\right|=\lim_{(x,y)\to(0,0)}\frac{|\varphi(xy)|}{\sqrt{x^2+y^2}}\leq\lim_{(x,y)\to(0,0)}\frac{C|xy|^{\mu}}{\sqrt{x^2+y^2}}\leq\lim_{(x,y)\to(0,0)}\frac{C|xy|^{\mu}}{\sqrt{2}|xy|^{\frac{1}{2}}}=\lim_{(x,y)\to(0,0)}\frac{C}{\sqrt{2}}|xy|^{\mu-\frac{1}{2}}=0$$

$$\Rightarrow\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}=0.\square$$

9.$f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=[f(x_0+\Delta x,y_0+\Delta y)-f(x_0+\Delta x,y_0)]+[f(x_0+\Delta x,y_0)-f(x_0,y_0)]$

因为$f_x(x_0,y_0)$存在,则$f(x_0+\Delta x,y_0)-f(x_0,y_0)=f_x(x_0,y_0)\Delta x+o(\Delta x)=f_x(x_0,y_0)\Delta x+o\left(\sqrt{(\Delta x)^2+(\Delta y)^2}\right)$

因为$f_y(x,y)$在$(x_0,y_0)$附近连续,于是存在$\theta\in(0,1)$,使得

$f(x_0+\Delta x,y_0+\Delta y)-f(x_0+\Delta x,y_0)=f_y(x_0+\Delta x,y_0+\theta\Delta y)\Delta y=f_y(x_0,y_0)\Delta y+o\left(\sqrt{(\Delta x)^2+(\Delta y)^2}\right)$

$\Rightarrow f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y+o\left(\sqrt{(\Delta x)^2+(\Delta y)^2}\right)$.

故$f$在$(x_0,y_0)$可微.□

11.定义在$\mathbb{R}^m$上的函数:

$$f(x)=\begin{cases}|x|^{\mu}\sin\dfrac{1}{|x|},x\neq 0\\ 0,x=0\end{cases}$$,其中$\mu$是常数,确定$\mu$的范围使得

(1)$f(x)$在$x=0$连续

(2)$f(x)$在$x=0$可微

(3)$f(x)$的偏导数在$x=0$处连续

解:(1)$\boldsymbol{x}\to\boldsymbol{0}$时,$|x|^{\mu}\sin\dfrac{1}{|x|}\sim|x|^{\mu-1}$,于是$\mu-1>0\Leftrightarrow\lim\limits_{(x,y)\to(0,0)}f(x)=0$.故$\mu>1$.

(2)$\boldsymbol{x}\to\boldsymbol{0}$时,$f(\Delta x)-f(0)=|\Delta x|^{\mu}\sin\dfrac{1}{|\Delta x|}=O\left(|\Delta x|^{\mu-1}\right)$,故

$\mu>2\Leftrightarrow O\left(|\Delta x|^{\mu-1}\right)=o\left(|\Delta x|\right)\Leftrightarrow f(x)$在$x=0$可微.故$\mu>2$.

(3)$\boldsymbol{x}=(x_1,\cdots,x_m),x\neq 0$时,$f(x)=(x_1^2+\cdots+x_m^2)^{\frac{\mu}{2}}\sin\dfrac{1}{(x_1^2+\cdots+x_m^2)^{\frac{1}{2}}}$

$$\frac{\partial f}{\partial x_i}(x_1,\cdots,x_m)=\frac{\partial f}{\partial x_i}\left[(x_1^2+\cdots+x_m^2)^{\frac{\mu}{2}}\sin\frac{1}{(x_1^2+\cdots+x_m^2)^{\frac{1}{2}}}\right]$$

$$=\frac{\mu}{2}(x_1^2+\cdots+x_m^2)^{\frac{\mu}{2}-1}(2x_i)\sin\frac{1}{(x_1^2+\cdots+x_m^2)^{\frac{1}{2}}}+(x_1^2+\cdots+x_m^2)^{\frac{\mu}{2}}\cos\frac{1}{(x_1^2+\cdots+x_m^2)^{\frac{1}{2}}}\left(-\frac{2x_i}{(x_1^2+\cdots+x_m^2)^{\frac{3}{2}}}\right)$$

$$=\mu(x_1^2+\cdots+x_m^2)^{\frac{\mu-2}{2}}\cdot x_i\sin\frac{1}{(x_1^2+\cdots+x_m^2)^{\frac{1}{2}}}-2(x_1^2+\cdots+x_m^2)^{\frac{\mu-3}{2}}\cdot x_i\cos\frac{1}{(x_1^2+\cdots+x_m^2)^{\frac{1}{2}}}$$

$$=\mu|x|^{\mu-2}\cdot x_i\sin\frac{1}{|x|}-2|x|^{\mu-3}\cdot x_i\cos\frac{1}{|x|}$$

$\boldsymbol{x}\to\boldsymbol{0}$时,$\dfrac{\partial f}{\partial x_i}(x_1,\cdots,x_m)=x_iO\left(|x|^{\mu-3}\right)+x_iO\left(|x|^{\mu-3}\right)$

于是$\mu\geq 3\Leftrightarrow\lim\limits_{x\to 0}\dfrac{\partial f}{\partial x_i}(x_1,\cdots,x_m)=0\Leftrightarrow f(x)$的偏导数在$x=0$处连续.故$\mu\geq 3$.□

$$1.f(x,y)=\begin{cases}\dfrac{x^2}{x^2+y^4},x\neq 0\\ 1,x=0\end{cases}$$

设方向$\boldsymbol{l}=(\cos\theta,\sin\theta)$

$$f(t\cos\theta,t\sin\theta)-1=\frac{t^2\cos^2\theta}{t^2\cos^2\theta+t^4\sin^4\theta}-1=\frac{1}{1+t^2\sin^2\theta\tan^2\theta}-1=-t^2\sin^2\theta\tan^2\theta+o(t^3)$$

于是对于任意给定的方向$\boldsymbol{l}=(\cos\theta,\sin\theta)$,$f$的方向导数都存在,为$0$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=k\sqrt{x}}}f(x,y)=\lim_{\substack{(x,y)\to(0,0)\\ y=k\sqrt{x}}}\frac{x^2}{x^2+k^4x^2}=\frac{1}{1+k^4}\neq 0(若k\neq 0),故f在(0,0)不连续,自然不可微.\square$$

$2.(2)f(x,y)=\arctan\dfrac{x}{y},P_0=(x_0,y_0)\ (y_0\neq 0)$

$\boldsymbol{l}$与$Ox$轴正向的夹角为$45°$.

则$\boldsymbol{l}=\left(\dfrac{\sqrt{2}}{2},-\dfrac{\sqrt{2}}{2}\right)$或$\left(\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right)$

记$\boldsymbol{l}_1=\left(\dfrac{\sqrt{2}}{2},-\dfrac{\sqrt{2}}{2}\right),\boldsymbol{l}_2=\left(\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right)$

$$\frac{\partial f}{\partial \boldsymbol{l}_1}(x_0,y_0)=\lim_{t\to 0}\frac{1}{t}\left[f\left(x_0+\frac{\sqrt{2}}{2}t,y_0-\frac{\sqrt{2}}{2}t\right)-f(x_0,y_0)\right]=\lim_{t\to 0}\frac{1}{t}\left[\arctan\frac{x_0+\frac{\sqrt{2}}{2}t}{y_0-\frac{\sqrt{2}}{2}t}-\arctan\frac{x_0}{y_0}\right]$$

$$=\lim_{t\to 0}\frac{1}{t}\arctan\left[\frac{\frac{x_0+\frac{\sqrt{2}}{2}t}{y_0-\frac{\sqrt{2}}{2}t}-\frac{x_0}{y_0}}{1+\frac{x_0+\frac{\sqrt{2}}{2}t}{y_0-\frac{\sqrt{2}}{2}t}\cdot\frac{x_0}{y_0}}\right]=\lim_{t\to 0}\frac{1}{t}\arctan\left[\frac{\left(x_0+\frac{\sqrt{2}}{2}t\right)\cdot y_0-\left(y_0-\frac{\sqrt{2}}{2}t\right)\cdot x_0}{\left(y_0-\frac{\sqrt{2}}{2}t\right)\cdot y_0+\left(x_0+\frac{\sqrt{2}}{2}t\right)\cdot x_0}\right]$$

$$=\lim_{t\to 0}\frac{1}{t}\arctan\left[\frac{\frac{\sqrt{2}}{2}t\cdot(x_0+y_0)}{x_0^2+y_0^2+\frac{\sqrt{2}}{2}t(x_0-y_0)}\right]\overset{y_0\neq 0\Rightarrow\frac{\frac{\sqrt{2}}{2}t\cdot(x_0+y_0)}{x_0^2+y_0^2+\frac{\sqrt{2}}{2}t(x_0-y_0)}\to 0}{=}\lim_{t\to 0}\frac{\frac{\frac{\sqrt{2}}{2}t\cdot(x_0+y_0)}{x_0^2+y_0^2+\frac{\sqrt{2}}{2}t(x_0-y_0)}}{t}$$

$$=\lim_{t\to 0}\frac{\frac{\sqrt{2}}{2}(x_0+y_0)}{x_0^2+y_0^2+\frac{\sqrt{2}}{2}t(x_0-y_0)}=\frac{x_0+y_0}{\sqrt{2}(x_0^2+y_0^2)}.$$

$$\frac{\partial f}{\partial \boldsymbol{l}_2}(x_0,y_0)=\lim_{t\to 0}\frac{1}{t}\left[f\left(x_0+\frac{\sqrt{2}}{2}t,y_0+\frac{\sqrt{2}}{2}t\right)-f(x_0,y_0)\right]=\lim_{t\to 0}\frac{1}{t}\left[\arctan\frac{x_0+\frac{\sqrt{2}}{2}t}{y_0+\frac{\sqrt{2}}{2}t}-\arctan\frac{x_0}{y_0}\right]$$

$$=\lim_{t\to 0}\frac{1}{t}\arctan\left[\frac{\frac{x_0+\frac{\sqrt{2}}{2}t}{y_0+\frac{\sqrt{2}}{2}t}-\frac{x_0}{y_0}}{1+\frac{x_0+\frac{\sqrt{2}}{2}t}{y_0+\frac{\sqrt{2}}{2}t}\cdot\frac{x_0}{y_0}}\right]=\lim_{t\to 0}\frac{1}{t}\arctan\left[\frac{\left(x_0+\frac{\sqrt{2}}{2}t\right)\cdot y_0-\left(y_0+\frac{\sqrt{2}}{2}t\right)\cdot x_0}{\left(y_0+\frac{\sqrt{2}}{2}t\right)\cdot y_0+\left(x_0+\frac{\sqrt{2}}{2}t\right)\cdot x_0}\right]$$

$$=\lim_{t\to 0}\frac{1}{t}\arctan\left[\frac{\frac{\sqrt{2}}{2}t\cdot(y_0-x_0)}{x_0^2+y_0^2+\frac{\sqrt{2}}{2}t(x_0+y_0)}\right]\overset{y_0\neq 0\Rightarrow\frac{\frac{\sqrt{2}}{2}t\cdot(y_0-x_0)}{x_0^2+y_0^2+\frac{\sqrt{2}}{2}t(x_0+y_0)}\to 0}{=}\lim_{t\to 0}\frac{\frac{\frac{\sqrt{2}}{2}t\cdot(y_0-x_0)}{x_0^2+y_0^2+\frac{\sqrt{2}}{2}t(x_0+y_0)}}{t}$$

$$=\lim_{t\to 0}\frac{\frac{\sqrt{2}}{2}t\cdot(y_0-x_0)}{x_0^2+y_0^2+\frac{\sqrt{2}}{2}t(x_0+y_0)}=\frac{y_0-x_0}{\sqrt{2}(x_0^2+y_0^2)}.$$

2.(3) $f(x,y,z)=x^2+2y^2+3z^2, P_0=(2,2,1)$

$$\boldsymbol{l}=\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

$$\frac{\partial f}{\partial \boldsymbol{l}}(x_0,y_0,z_0)=\lim_{t\to 0}\frac{1}{t}\left[f\left(x_0+\frac{1}{\sqrt{3}}t,y_0-\frac{1}{\sqrt{3}}t,z_0+\frac{1}{\sqrt{3}}t\right)-f(x_0,y_0,z_0)\right]$$

$$=\lim_{t\to 0}\frac{1}{t}\left[\left(x_0+\frac{1}{\sqrt{3}}t\right)^2+2\left(y_0-\frac{1}{\sqrt{3}}t\right)^2+3\left(z_0+\frac{1}{\sqrt{3}}t\right)^2-x_0^2-2y_0^2-3z_0^2\right]$$

$$=\lim_{t\to 0}\frac{1}{t}\left[\frac{2t}{\sqrt{3}}(x_0-2y_0+3z_0)+2t^2\right]=\frac{2}{\sqrt{3}}(x_0-2y_0+3z_0)$$

3. $\nabla f(P_0)=(f_x(P_0),f_y(P_0))$

$$\begin{cases}\dfrac{\partial f}{\partial \boldsymbol{l}_1}(P_0)=1\\ \dfrac{\partial f}{\partial \boldsymbol{l}_2}(P_0)=-1\end{cases}\Rightarrow\begin{cases}\boldsymbol{l}_1\cdot\nabla f(P_0)=1\\ \boldsymbol{l}_2\cdot\nabla f(P_0)=-1\end{cases}\Rightarrow\begin{cases}f_x(P_0)+f_y(P_0)=\sqrt{2}\\ f_x(P_0)-f_y(P_0)=-\sqrt{2}\end{cases}\Rightarrow\begin{cases}f_x(P_0)=0\\ f_y(P_0)=\sqrt{2}\end{cases}$$

$$\Rightarrow\nabla f(P_0)=\left(0,\sqrt{2}\right)$$

4.设 $\boldsymbol{l}_i=(\cos\theta_i,\sin\theta_i)$,补充定义 $\boldsymbol{l}_{n+1}=\boldsymbol{l}_1$

于是 $\boldsymbol{l}_{i+1}$ 与 $\boldsymbol{l}_i$ 夹角为 $\frac{2\pi}{n}\Rightarrow\boldsymbol{l}_i\cdot\boldsymbol{l}_{i+1}=\cos\frac{2\pi}{n}\Rightarrow\cos\theta_i\cos\theta_{i+1}+\sin\theta_i\sin\theta_{i+1}=\cos\frac{2\pi}{n}$

$\Rightarrow\cos(\theta_i-\theta_{i+1})=\cos\frac{2\pi}{n}$,若 $\{\boldsymbol{l}_i\}_{i=1}^n$ 两两不同,不妨令 $\theta_{i+1}-\theta_i=\frac{2\pi}{n},\forall i.$

于是 $\theta_i=\theta_1+\frac{2\pi}{n}(i-1),\forall i.$

于是 $$\sum_{i=1}^n\frac{\partial f}{\partial\boldsymbol{l}_i}(P_0)=\sum_{i=1}^n\nabla f(P_0)\cdot\boldsymbol{l}_i=\nabla f(P_0)\cdot\left(\sum_{i=1}^n\boldsymbol{l}_i\right)=\nabla f(P_0)\cdot\left(\sum_{i=1}^n\cos(\theta_i),\sum_{i=1}^n\sin(\theta_i)\right)$$

$$=\nabla f(P_0)\cdot\left(\sum_{i=1}^n\cos\left(\theta_1+\frac{2\pi}{n}(i-1)\right),\sum_{i=1}^n\sin\left(\theta_1+\frac{2\pi}{n}(i-1)\right)\right)$$

$$=\nabla f(P_0)\cdot\left(\frac{\sum_{i=1}^n\sin\frac{\pi}{n}\cos\left(\theta_1+\frac{2\pi}{n}(i-1)\right)}{\sin\frac{\pi}{n}},\frac{\sum_{i=1}^n\sin\frac{\pi}{n}\sin\left(\theta_1+\frac{2\pi}{n}(i-1)\right)}{\sin\frac{\pi}{n}}\right)$$

$$=\nabla f(P_0)\cdot\left(\frac{\sum_{i=1}^n\left[\sin\left(\theta_1+\frac{\pi}{n}(2i-1)\right)-\sin\left(\theta_1+\frac{\pi}{n}(2i-3)\right)\right]}{2\sin\frac{\pi}{n}},\frac{\sum_{i=1}^n\left[\cos\left(\theta_1+\frac{\pi}{n}(2i-1)\right)-\cos\left(\theta_1+\frac{\pi}{n}(2i-3)\right)\right]}{2\sin\frac{\pi}{n}}\right)$$

$$=\nabla f(P_0)\cdot(0,0)=0.\square$$

7. (1) $\frac{\partial f}{\partial \boldsymbol{l}}(x_0) \overset{?}{=} \sum_{i=1}^{m} \frac{\partial f}{\partial \boldsymbol{e}_i}(x_0)\cos\langle \boldsymbol{l}, \boldsymbol{e}_i \rangle$

$\frac{\partial f}{\partial \boldsymbol{l}}(x_0) = \nabla f(x_0) \cdot \boldsymbol{l}$

$$\sum_{i=1}^{m} \frac{\partial f}{\partial \boldsymbol{e}_i}(x_0)\cos\langle \boldsymbol{l}, \boldsymbol{e}_i \rangle = \sum_{i=1}^{m} \nabla f(x_0) \cdot \boldsymbol{e}_i \cos\langle \boldsymbol{l}, \boldsymbol{e}_i \rangle = \nabla f(x_0) \cdot \sum_{i=1}^{m} \boldsymbol{e}_i \cos\langle \boldsymbol{l}, \boldsymbol{e}_i \rangle$$

只需要验证: $\boldsymbol{l} = \sum_{i=1}^{m} \boldsymbol{e}_i \cos\langle \boldsymbol{l}, \boldsymbol{e}_i \rangle$

由于 $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_m$ 是 $\mathbb{R}^m$ 中 $m$ 个两两正交的单位向量,故构成 $\mathbb{R}^m$ 的一组基.

于是 $\boldsymbol{l}$ 可以表示为 $a_1\boldsymbol{e}_1 + \cdots + a_m\boldsymbol{e}_m$

于是 $\sum_{i=1}^{m} \boldsymbol{e}_i \cos\langle \boldsymbol{l}, \boldsymbol{e}_i \rangle = \sum_{i=1}^{m} \boldsymbol{e}_i \cos\langle a_1\boldsymbol{e}_1 + \cdots + a_m\boldsymbol{e}_m, \boldsymbol{e}_i \rangle$

$$= \sum_{i=1}^{m} \boldsymbol{e}_i \frac{(a_1\boldsymbol{e}_1 + \cdots + a_m\boldsymbol{e}_m) \cdot \boldsymbol{e}_i}{|a_1\boldsymbol{e}_1 + \cdots + a_m\boldsymbol{e}_m| \cdot |\boldsymbol{e}_i|} = \sum_{i=1}^{m} \boldsymbol{e}_i [(a_1\boldsymbol{e}_1 + \cdots + a_m\boldsymbol{e}_m) \cdot \boldsymbol{e}_i]$$

$$= \sum_{i=1}^{m} \boldsymbol{e}_i a_i = \boldsymbol{l}. \square$$

(2) $|\nabla f(x_0)| \overset{?}{=} \sqrt{\sum_{i=1}^{m} \left|\frac{\partial f}{\partial \boldsymbol{e}_i}(x_0)\right|^2}$

$$\sum_{i=1}^{m} \left|\frac{\partial f}{\partial \boldsymbol{e}_i}(x_0)\right|^2 = \sum_{i=1}^{m} |\nabla f(x_0) \cdot \boldsymbol{e}_i|^2 = \sum_{i=1}^{m} |\nabla f(x_0) \cdot \boldsymbol{e}_i|^2$$

设 $\nabla f(x_0) = b_1\boldsymbol{e}_1 + \cdots + b_m\boldsymbol{e}_m$, 于是 $|\nabla f(x_0)|^2 = b_1^2 + \cdots + b_m^2$

$$\sum_{i=1}^{m} |\nabla f(x_0) \cdot \boldsymbol{e}_i|^2 = \sum_{i=1}^{m} |(b_1\boldsymbol{e}_1 + \cdots + b_m\boldsymbol{e}_m) \cdot \boldsymbol{e}_i|^2 = \sum_{i=1}^{m} b_i^2 = |\nabla f(x_0)|^2. \square$$

11.**Pf:** $f$ 在 $D\subseteq\mathbb{R}^m$ 上可微, $\nabla f(x)=0$, $\forall x\in D$.

我们考虑区域 $D$ 的内点 $D^\circ$, 这显然是一个开区域.

显然 $D$ 没有孤点, 于是 $D=\partial D\sqcup D^\circ$. 任取 $x_0\in D^\circ$.

如果 $f$ 在开区域 $D^\circ$ 上面恒取常值 $f(x_0)$

那么由于 $f$ 在 $D$ 上可微, 故 $f$ 在 $D$ 上连续.

对于任意 $x\in\partial D$, $x$ 都是 $D^\circ$ 的极限点, 故存在 $D^\circ$ 中一列点 $\{x_n\}_{n\geq 1}\to x$

故 $f(x)=\lim\limits_{n\to\infty}f(x_n)=f(x_0)$.

于是我们不妨设 $D$ 是一个开区域.

任取 $x_0\in D$, 都存在一个开邻域 $B_\delta(x_0)=\{x:|x-x_0|<\delta\}\subset D$, 这是一个凸集.

于是 $\forall x\in B_\delta(x_0)-\{x_0\}$, 由微分中值定理: 存在 $x$ 与 $x_0$ 连线段上的一个点 $\xi$(显然 $\xi\in B_\delta(x_0)\subset D$), 使得

$f(x)-f(x_0)=\nabla f(\xi)\,(x-x_0)=0$, 于是 $f(B_\delta(x_0))=f(x_0)$.

考虑集合 $E=\{x\in D: f(x)=f(x_0)\}$, 显然 $x_0\in E\Rightarrow E\neq\varnothing$.

我们下面证明 $E$ 相对 $D$ 既开又闭.

①$E$ 开：对于任意 $x_1\in E\subset D$, 由上面关于 $x_0$ 的讨论可知: 存在 $x_1$ 的邻域 $B_\delta(x_1)\subset D$, $f(B_\delta(x_1))=f(x_1)=f(x_0)$.

$\Rightarrow B_\delta(x_1)\subset E$. 于是 $D$ 是开集.

②$E$ 闭：只需证明 $E$ 的极限点都在 $E$ 中. 若 $x_2\in D$, 满足对于任意给定的 $\epsilon>0$, $B_\epsilon(x_2)\cap E\neq\varnothing$.

故存在 $E$ 中一列点列 $\{y_n\}_{n\geq 1}\to x_2$, 于是由 $f$ 连续性可知: $f(x_2)=\lim\limits_{n\to\infty}f(y_n)=f(x_0)$. 故 $x_2\in E$. 于是 $E$ 是闭集.

由于在欧式空间中, 相对集合 $D$ 既开又闭的集合只有全集和空集, 又因为 $E\neq\varnothing$, 故 $E=D$.

于是 $f(x)$ 在 $D$ 上恒取常值. $\square$

1. $\dfrac{\partial w}{\partial u}=\dfrac{\partial f(x(u,v),y(u,v),z(u,v))}{\partial u}$

$$=f_x(x(u,v),y(u,v),z(u,v))\frac{\partial x(u,v)}{\partial u}+f_y(x(u,v),y(u,v),z(u,v))\frac{\partial y(u,v)}{\partial u}+f_z(x(u,v),y(u,v),z(u,v))\frac{\partial z(u,v)}{\partial u}$$

$$\frac{\partial w}{\partial v}=f_x(x(u,v),y(u,v),z(u,v))\frac{\partial x(u,v)}{\partial v}+f_y(x(u,v),y(u,v),z(u,v))\frac{\partial y(u,v)}{\partial v}+f_z(x(u,v),y(u,v),z(u,v))\frac{\partial z(u,v)}{\partial v}$$

2.(4) $u=yf\left(y+\frac{y^3}{x^2}\right)$

$$\frac{\partial u}{\partial y}=\frac{\partial\left[yf\left(y+\frac{y^3}{x^2}\right)\right]}{\partial y}=f\left(y+\frac{y^3}{x^2}\right)+y\frac{\partial f\left(y+\frac{y^3}{x^2}\right)}{\partial\left(y+\frac{y^3}{x^2}\right)}\frac{\partial\left(y+\frac{y^3}{x^2}\right)}{\partial y}=f\left(y+\frac{y^3}{x^2}\right)+yf'\left(y+\frac{y^3}{x^2}\right)\left(1+\frac{3y^2}{x^2}\right)$$

$$\frac{\partial u}{\partial x}=\frac{\partial\left[yf\left(y+\frac{y^3}{x^2}\right)\right]}{\partial x}=y\frac{\partial f\left(y+\frac{y^3}{x^2}\right)}{\partial\left(y+\frac{y^3}{x^2}\right)}\frac{\partial\left(y+\frac{y^3}{x^2}\right)}{\partial x}=yf'\left(y+\frac{y^3}{x^2}\right)\left(-2\frac{y^3}{x^3}\right)=\left(-2\frac{y^4}{x^3}\right)f'\left(y+\frac{y^3}{x^2}\right)$$

2.(6) $u=\sqrt{x^2+y^2}\,f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x}=\frac{\partial\left[\sqrt{x^2+y^2}\,f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)\right]}{\partial x}$$

$$=\frac{\partial\sqrt{x^2+y^2}}{\partial x}f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)+\sqrt{x^2+y^2}\frac{\partial f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)}{\partial x}$$

$$=\frac{x}{\sqrt{x^2+y^2}}f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)+\sqrt{x^2+y^2}\frac{\partial f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)}{\partial\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)}\frac{\partial\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)}{\partial x}$$

$$=\frac{x}{\sqrt{x^2+y^2}}f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)+\sqrt{x^2+y^2}f'\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)\left[\frac{\partial\left(\ln\sqrt{x^2+y^2}\right)}{\partial x}+\frac{\partial\left(\arctan\frac{y}{x}\right)}{\partial x}\right]$$

$$=\frac{x}{\sqrt{x^2+y^2}}f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)+\sqrt{x^2+y^2}f'\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)\left[\frac{x}{x^2+y^2}+\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)\right]$$

$$=\frac{x}{\sqrt{x^2+y^2}}f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)+\frac{x-y}{\sqrt{x^2+y^2}}f'\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y}=\frac{\partial\left[\sqrt{x^2+y^2}\,f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)\right]}{\partial y}$$

$$=\frac{\partial\sqrt{x^2+y^2}}{\partial y}f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)+\sqrt{x^2+y^2}\frac{\partial f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)}{\partial y}$$

$$=\frac{y}{\sqrt{x^2+y^2}}f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)+\sqrt{x^2+y^2}\frac{\partial f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)}{\partial\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)}\frac{\partial\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)}{\partial y}$$

$$=\frac{y}{\sqrt{x^2+y^2}}f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)+\sqrt{x^2+y^2}f'\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)\left[\frac{\partial\left(\ln\sqrt{x^2+y^2}\right)}{\partial y}+\frac{\partial\left(\arctan\frac{y}{x}\right)}{\partial y}\right]$$

$$=\frac{y}{\sqrt{x^2+y^2}}f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)+\sqrt{x^2+y^2}f'\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)\left[\frac{y}{x^2+y^2}+\frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x}\right]$$

$$=\frac{y}{\sqrt{x^2+y^2}}f\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)+\frac{x+y}{\sqrt{x^2+y^2}}f'\left(\ln\sqrt{x^2+y^2}+\arctan\frac{y}{x}\right)$$

2.(7) $u=xf\left(\frac{z}{x}\right)+yf\left(\frac{z}{y}\right)$

$$\frac{\partial u}{\partial x}=\frac{\partial\left[xf\left(\frac{z}{x}\right)+yf\left(\frac{z}{y}\right)\right]}{\partial x}=\frac{\partial\left[xf\left(\frac{z}{x}\right)\right]}{\partial x}=f\left(\frac{z}{x}\right)+x\frac{\partial f\left(\frac{z}{x}\right)}{\partial x}=f\left(\frac{z}{x}\right)+x\frac{\partial f\left(\frac{z}{x}\right)}{\partial\left(\frac{z}{x}\right)}\frac{\partial\left(\frac{z}{x}\right)}{\partial x}=f\left(\frac{z}{x}\right)+xf'\left(\frac{z}{x}\right)\left(-\frac{z}{x^2}\right)=f\left(\frac{z}{x}\right)-\frac{z}{x}f'\left(\frac{z}{x}\right)$$

$$\frac{\partial u}{\partial y}=\frac{\partial\left[xf\left(\frac{z}{x}\right)+yf\left(\frac{z}{y}\right)\right]}{\partial y}=\frac{\partial\left[yf\left(\frac{z}{y}\right)\right]}{\partial y}=f\left(\frac{z}{y}\right)+y\frac{\partial f\left(\frac{z}{y}\right)}{\partial y}=f\left(\frac{z}{y}\right)+y\frac{\partial f\left(\frac{z}{y}\right)}{\partial\left(\frac{z}{y}\right)}\frac{\partial\left(\frac{z}{y}\right)}{\partial y}=f\left(\frac{z}{y}\right)+yf'\left(\frac{z}{y}\right)\left(-\frac{z}{y^2}\right)=f\left(\frac{z}{y}\right)-\frac{z}{y}f'\left(\frac{z}{y}\right)$$

$$\frac{\partial u}{\partial z}=\frac{\partial\left[xf\left(\frac{z}{x}\right)+yf\left(\frac{z}{y}\right)\right]}{\partial z}=x\frac{\partial\left[f\left(\frac{z}{x}\right)\right]}{\partial z}+y\frac{\partial\left[f\left(\frac{z}{y}\right)\right]}{\partial z}=f'\left(\frac{z}{x}\right)+f'\left(\frac{z}{y}\right).$$

3.(1) $u=f(ax+by,bx-ay)$

$$\frac{\partial u}{\partial x}=\frac{\partial f(ax+by,bx-ay)}{\partial x}=f_1(ax+by,bx-ay)\frac{\partial(ax+by)}{\partial x}+f_2(ax+by,bx-ay)\frac{\partial(bx-ay)}{\partial x}$$

$$=af_1(ax+by,bx-ay)+bf_2(ax+by,bx-ay)$$

$$\frac{\partial u}{\partial y}=\frac{\partial f(ax+by,bx-ay)}{\partial y}=f_1(ax+by,bx-ay)\frac{\partial(ax+by)}{\partial y}+f_2(ax+by,bx-ay)\frac{\partial(bx-ay)}{\partial y}$$

$$=bf_1(ax+by,bx-ay)-af_2(ax+by,bx-ay)$$

3.(2) $u=f\left(x^2+y^2,\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x}=\frac{\partial f\left(x^2+y^2,\frac{y}{x}\right)}{\partial x}=f_1\left(x^2+y^2,\frac{y}{x}\right)\frac{\partial(x^2+y^2)}{\partial x}+f_2\left(x^2+y^2,\frac{y}{x}\right)\frac{\partial\left(\frac{y}{x}\right)}{\partial x}$$

$$=2xf_1\left(x^2+y^2,\frac{y}{x}\right)-\frac{y}{x^2}f_2\left(x^2+y^2,\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y}=\frac{\partial f\left(x^2+y^2,\frac{y}{x}\right)}{\partial y}=f_1\left(x^2+y^2,\frac{y}{x}\right)\frac{\partial(x^2+y^2)}{\partial y}+f_2\left(x^2+y^2,\frac{y}{x}\right)\frac{\partial\left(\frac{y}{x}\right)}{\partial y}$$

$$=2yf_1\left(x^2+y^2,\frac{y}{x}\right)+\frac{1}{x}f_2\left(x^2+y^2,\frac{y}{x}\right)$$

3.(5) $u=f(xy^2,yz^2,zx^2)$

$$\frac{\partial u}{\partial x}=\frac{\partial f(xy^2,yz^2,zx^2)}{\partial x}=f_1(xy^2,yz^2,zx^2)\frac{\partial(xy^2)}{\partial x}+f_2(xy^2,yz^2,zx^2)\frac{\partial(yz^2)}{\partial x}+f_3(xy^2,yz^2,zx^2)\frac{\partial(zx^2)}{\partial x}$$

$$=y^2f_1(xy^2,yz^2,zx^2)+2zxf_3(xy^2,yz^2,zx^2)$$

$$\frac{\partial u}{\partial y}=\frac{\partial f(xy^2,yz^2,zx^2)}{\partial y}=f_1(xy^2,yz^2,zx^2)\frac{\partial(xy^2)}{\partial y}+f_2(xy^2,yz^2,zx^2)\frac{\partial(yz^2)}{\partial y}+f_3(xy^2,yz^2,zx^2)\frac{\partial(zx^2)}{\partial y}$$

$$=2xyf_1(xy^2,yz^2,zx^2)+z^2f_2(xy^2,yz^2,zx^2)$$

$$\frac{\partial u}{\partial z}=\frac{\partial f(xy^2,yz^2,zx^2)}{\partial z}=f_1(xy^2,yz^2,zx^2)\frac{\partial(xy^2)}{\partial z}+f_2(xy^2,yz^2,zx^2)\frac{\partial(yz^2)}{\partial z}+f_3(xy^2,yz^2,zx^2)\frac{\partial(zx^2)}{\partial z}$$

$$=2yzf_2(xy^2,yz^2,zx^2)+x^2f_3(xy^2,yz^2,zx^2)$$

3.(6) $u=f(x^2+y^2,x^2-y^2,2xy)$

$$\frac{\partial u}{\partial x}=\frac{\partial f(x^2+y^2,x^2-y^2,2xy)}{\partial x}=f_1(x^2+y^2,x^2-y^2,2xy)\frac{\partial(x^2+y^2)}{\partial x}+f_2(x^2+y^2,x^2-y^2,2xy)\frac{\partial(x^2-y^2)}{\partial x}+f_3(x^2+y^2,x^2-y^2,2xy)\frac{\partial(2xy)}{\partial x}$$

$$=2xf_1(x^2+y^2,x^2-y^2,2xy)+2xf_2(x^2+y^2,x^2-y^2,2xy)+2yf_3(x^2+y^2,x^2-y^2,2xy)$$

$$\frac{\partial u}{\partial y}=\frac{\partial f(x^2+y^2,x^2-y^2,2xy)}{\partial y}=f_1(x^2+y^2,x^2-y^2,2xy)\frac{\partial(x^2+y^2)}{\partial y}+f_2(x^2+y^2,x^2-y^2,2xy)\frac{\partial(x^2-y^2)}{\partial y}+f_3(x^2+y^2,x^2-y^2,2xy)\frac{\partial(2xy)}{\partial y}$$

$$=2yf_1(x^2+y^2,x^2-y^2,2xy)-2yf_2(x^2+y^2,x^2-y^2,2xy)+2xf_3(x^2+y^2,x^2-y^2,2xy)$$

4.(3) $(1+x^2)\dfrac{\partial u}{\partial x}+xy\dfrac{\partial u}{\partial y}=0$,其中 $u=f\left(\dfrac{y^2}{1+x^2}\right)$

$$\frac{\partial u}{\partial x}=\frac{\partial f\left(\frac{y^2}{1+x^2}\right)}{\partial x}=\frac{\partial f\left(\frac{y^2}{1+x^2}\right)}{\partial\left(\frac{y^2}{1+x^2}\right)}\frac{\partial\left(\frac{y^2}{1+x^2}\right)}{\partial x}=f'\left(\frac{y^2}{1+x^2}\right)\left(-\frac{2xy^2}{(1+x^2)^2}\right)=-\frac{2xy^2f'\left(\frac{y^2}{1+x^2}\right)}{(1+x^2)^2}$$

$$\frac{\partial u}{\partial y}=\frac{\partial f\left(\frac{y^2}{1+x^2}\right)}{\partial y}=\frac{\partial f\left(\frac{y^2}{1+x^2}\right)}{\partial\left(\frac{y^2}{1+x^2}\right)}\frac{\partial\left(\frac{y^2}{1+x^2}\right)}{\partial y}=f'\left(\frac{y^2}{1+x^2}\right)\frac{2y}{1+x^2}=\frac{2yf'\left(\frac{y^2}{1+x^2}\right)}{1+x^2}$$

于是 $(1+x^2)\dfrac{\partial u}{\partial x}+xy\dfrac{\partial u}{\partial y}=-(1+x^2)\dfrac{2xy^2f'\left(\frac{y^2}{1+x^2}\right)}{(1+x^2)^2}+xy\dfrac{2yf'\left(\frac{y^2}{1+x^2}\right)}{1+x^2}=0.$

4.(5) $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=u+x^2+y^2$,其中 $u=xf\left(\dfrac{y}{x}\right)+x^2+y^2$

$$\frac{\partial u}{\partial x}=\frac{\partial\left[xf\left(\frac{y}{x}\right)+x^2+y^2\right]}{\partial x}=\frac{\partial\left[xf\left(\frac{y}{x}\right)\right]}{\partial x}+\frac{\partial(x^2+y^2)}{\partial x}=f\left(\frac{y}{x}\right)+x\frac{\partial f\left(\frac{y}{x}\right)}{\partial x}+2x$$

$$=f\left(\frac{y}{x}\right)+x\frac{\partial f\left(\frac{y}{x}\right)}{\partial\left(\frac{y}{x}\right)}\frac{\partial\left(\frac{y}{x}\right)}{\partial x}+2x=f\left(\frac{y}{x}\right)+xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)+2x=f\left(\frac{y}{x}\right)-\frac{y}{x}f'\left(\frac{y}{x}\right)+2x$$

$$\frac{\partial u}{\partial y}=\frac{\partial\left[xf\left(\frac{y}{x}\right)+x^2+y^2\right]}{\partial y}=\frac{\partial\left[xf\left(\frac{y}{x}\right)\right]}{\partial y}+\frac{\partial(x^2+y^2)}{\partial y}=x\frac{\partial f\left(\frac{y}{x}\right)}{\partial\left(\frac{y}{x}\right)}\frac{\partial\left(\frac{y}{x}\right)}{\partial y}+2y=f'\left(\frac{y}{x}\right)+2y$$

于是 $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=x\left[f\left(\dfrac{y}{x}\right)-\dfrac{y}{x}f'\left(\dfrac{y}{x}\right)+2x\right]+y\left[f'\left(\dfrac{y}{x}\right)+2y\right]=xf\left(\dfrac{y}{x}\right)+2x^2+2y^2=u+x^2+y^2.$

5.$F$是二元可微函数,验证:

5.(3) $\sqrt{x}\frac{\partial u}{\partial x}+\sqrt{y}\frac{\partial u}{\partial y}+\sqrt{z}\frac{\partial u}{\partial z}=0$,其中 $u=F(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})$

$$\frac{\partial u}{\partial x}=\frac{\partial F(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})}{\partial x}=F_1(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})\frac{\partial(\sqrt{x}-\sqrt{y})}{\partial x}+F_2(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})\frac{\partial(\sqrt{z}-\sqrt{x})}{\partial x}$$

$$=\frac{1}{2\sqrt{x}}F_1(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})-\frac{1}{2\sqrt{x}}F_2(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})$$

$$\frac{\partial u}{\partial y}=\frac{\partial F(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})}{\partial y}=F_1(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})\frac{\partial(\sqrt{x}-\sqrt{y})}{\partial y}+F_2(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})\frac{\partial(\sqrt{z}-\sqrt{x})}{\partial y}=-\frac{1}{2\sqrt{y}}F_1(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})$$

$$\frac{\partial u}{\partial z}=\frac{\partial F(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})}{\partial z}=F_1(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})\frac{\partial(\sqrt{x}-\sqrt{y})}{\partial z}+F_2(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})\frac{\partial(\sqrt{z}-\sqrt{x})}{\partial z}=\frac{1}{2\sqrt{z}}F_2(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})$$

$$\Rightarrow\sqrt{x}\frac{\partial u}{\partial x}+\sqrt{y}\frac{\partial u}{\partial y}+\sqrt{z}\frac{\partial u}{\partial z}$$

$$=\sqrt{x}\left[\frac{1}{2\sqrt{x}}F_1(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})-\frac{1}{2\sqrt{x}}F_2(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})\right]+\sqrt{y}\left[-\frac{1}{2\sqrt{y}}F_1(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})\right]+\sqrt{z}\left[\frac{1}{2\sqrt{z}}F_2(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})\right]$$

$$=\frac{1}{2}F_1(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})-\frac{1}{2}F_2(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})-\frac{1}{2}F_1(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})+\frac{1}{2}F_2(\sqrt{x}-\sqrt{y},\sqrt{z}-\sqrt{x})=0$$

5.(6) $(bz-cy)\frac{\partial u}{\partial x}+(cx-az)\frac{\partial u}{\partial y}+(ay-bx)\frac{\partial u}{\partial z}=0$,其中 $u=F(ax+by+cz,x^2+y^2+z^2)$

$$\frac{\partial u}{\partial x}=\frac{\partial F(ax+by+cz,x^2+y^2+z^2)}{\partial x}=F_1(ax+by+cz,x^2+y^2+z^2)\frac{\partial(ax+by+cz)}{\partial x}+F_2(ax+by+cz,x^2+y^2+z^2)\frac{\partial(x^2+y^2+z^2)}{\partial x}$$

$$=aF_1(ax+by+cz,x^2+y^2+z^2)+2xF_2(ax+by+cz,x^2+y^2+z^2)$$

$$\frac{\partial u}{\partial y}=\frac{\partial F(ax+by+cz,x^2+y^2+z^2)}{\partial y}=F_1(ax+by+cz,x^2+y^2+z^2)\frac{\partial(ax+by+cz)}{\partial y}+F_2(ax+by+cz,x^2+y^2+z^2)\frac{\partial(x^2+y^2+z^2)}{\partial y}$$

$$=bF_1(ax+by+cz,x^2+y^2+z^2)+2yF_2(ax+by+cz,x^2+y^2+z^2)$$

$$\frac{\partial u}{\partial z}=\frac{\partial F(ax+by+cz,x^2+y^2+z^2)}{\partial z}=F_1(ax+by+cz,x^2+y^2+z^2)\frac{\partial(ax+by+cz)}{\partial z}+F_2(ax+by+cz,x^2+y^2+z^2)\frac{\partial(x^2+y^2+z^2)}{\partial z}$$

$$=cF_1(ax+by+cz,x^2+y^2+z^2)+2zF_2(ax+by+cz,x^2+y^2+z^2)$$

$$\Rightarrow(bz-cy)\frac{\partial u}{\partial x}+(cx-az)\frac{\partial u}{\partial y}+(ay-bx)\frac{\partial u}{\partial z}$$

$$=(bz-cy)[aF_1(ax+by+cz,x^2+y^2+z^2)+2xF_2(ax+by+cz,x^2+y^2+z^2)]$$

$$+(cx-az)[bF_1(ax+by+cz,x^2+y^2+z^2)+2yF_2(ax+by+cz,x^2+y^2+z^2)]$$

$$+(ay-bx)[cF_1(ax+by+cz,x^2+y^2+z^2)+2zF_2(ax+by+cz,x^2+y^2+z^2)]$$

$$=(abz-acy+bcx-baz+cay-cbx)F_1(ax+by+cz,x^2+y^2+z^2)+(2bxz-2cxy+2cxy-2ayz+2ayz-2bxz)F_2(ax+by+cz,x^2+y^2+z^2)=0.$$

6.("$\Rightarrow$":) 若有 $f(\lambda x)=\lambda^\mu f(x), x=(x_1,\cdots,x_m)\in\mathbb{R}^m-\{0\}$ 两边对 $\lambda$ 求导,得到

$$\frac{\partial f(\lambda x)}{\partial\lambda}=\frac{\partial[\lambda^\mu f(x)]}{\partial\lambda}=\mu\lambda^{\mu-1}f(x)$$

$$\frac{\partial f(\lambda x)}{\partial\lambda}=\sum_{i=1}^m\frac{\partial f(\lambda x)}{\partial(\lambda x_i)}\frac{\partial(\lambda x_i)}{\partial\lambda}=\sum_{i=1}^m x_i f_1(\lambda x)=\mu\lambda^{\mu-1}f(x)$$

取 $\lambda=1\Rightarrow\sum_{i=1}^m x_i\frac{\partial f(x)}{\partial x_i}=\sum_{i=1}^m x_i f_1(x)=\mu f(x)$.

("$\Leftarrow$":) 若有 $\sum_{i=1}^m x_i\frac{\partial f(x)}{\partial x_i}=\sum_{i=1}^m x_i f_1(x)=\mu f(x), \forall x\in\mathbb{R}^m-\{0\}$.那么对于任意 $\lambda>0$,有

$$\mu\lambda^{\mu-1}f(x)=\frac{\partial[\lambda^\mu f(x)]}{\partial\lambda}=\frac{\partial f(\lambda x)}{\partial\lambda}=\sum_{i=1}^m\frac{\partial f(\lambda x)}{\partial(\lambda x_i)}\frac{\partial(\lambda x_i)}{\partial\lambda}=\sum_{i=1}^m x_i f_1(\lambda x)=\frac{1}{\lambda}\sum_{i=1}^m(\lambda x_i)f_1(\lambda x)=\frac{\mu}{\lambda}f(\lambda x)$$

$\Rightarrow f(\lambda x)=\lambda^\mu f(x)$.□