

对于方向 $\mathbf{l} = (\cos\theta, \sin\theta), t > 0$.

$$3.(1) f(x, y) = \frac{ax + by}{\sqrt{x^2 + y^2}}$$

$$f(t \cos\theta, t \sin\theta) = \frac{at \cos\theta + bt \sin\theta}{|t|} = a \cos\theta + b \sin\theta$$

于是 f 沿方向 $\mathbf{l} = (\cos\theta, \sin\theta)$ 的极限为 $\lim_{t \rightarrow 0^+} f(t \cos\theta, t \sin\theta) = a \cos\theta + b \sin\theta$

$$3.(2) f(x, y) = \frac{\sin(xy)}{x^2 + y^2}$$

$$f(t \cos\theta, t \sin\theta) = \frac{\sin(t^2 \sin\theta \cos\theta)}{t^2}$$

于是 f 沿方向 $\mathbf{l} = (\cos\theta, \sin\theta)$ 的极限为 $\lim_{t \rightarrow 0^+} \frac{\sin(t^2 \sin\theta \cos\theta)}{t^2} = \sin\theta \cos\theta$.

$$3.(3) f(x, y) = \frac{x^2 y^2}{x^2 y^2 + (x - y)^2}$$

$$f(t \cos\theta, t \sin\theta) = \frac{t^2 \sin^2\theta \cos^2\theta}{t^2 \sin^2\theta \cos^2\theta + (t \cos\theta - t \sin\theta)^2}$$

$$= \frac{t^2 \sin^2\theta \cos^2\theta}{t^2 \sin^2\theta \cos^2\theta + t^2 - 2t^2 \sin\theta \cos\theta}$$

$$= \frac{\sin^2\theta \cos^2\theta}{\sin^2\theta \cos^2\theta - 2\sin\theta \cos\theta + 1}$$

于是 f 沿方向 $\mathbf{l} = (\cos\theta, \sin\theta)$ 的极限为 $\lim_{t \rightarrow 0^+} f(t \cos\theta, t \sin\theta) = \frac{\sin^2\theta \cos^2\theta}{\sin^2\theta \cos^2\theta - 2\sin\theta \cos\theta + 1}$

$$3.(4) f(x, y) = \frac{|x|^\alpha |y|^\beta}{x^2 + y^4} \quad (\alpha > 0, \beta > 0)$$

$$f(t \cos\theta, t \sin\theta) = \frac{|t \cos\theta|^\alpha |t \sin\theta|^\beta}{(t \cos\theta)^2 + (t \sin\theta)^4} = t^{\alpha+\beta-2} \frac{|\cos\theta|^\alpha |\sin\theta|^\beta}{\cos^2\theta + t^2 \sin^4\theta}$$

若 $\theta = 0, \frac{\pi}{2}, \pi$ 或 $\frac{3\pi}{2}$, 则 $f(t \cos\theta, t \sin\theta) = 0$

若 $\theta \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$,

① 若 $\alpha + \beta > 2$, 则 $f(t \cos\theta, t \sin\theta) = 0$

② 若 $\alpha + \beta = 2$, 则 $f(t \cos\theta, t \sin\theta) = \frac{|\cos\theta|^\alpha |\sin\theta|^\beta}{\cos^2\theta + t^2 \sin^4\theta}$

$$\lim_{t \rightarrow 0^+} f(t \cos\theta, t \sin\theta) = \frac{|\cos\theta|^\alpha |\sin\theta|^\beta}{\cos^2\theta} = |\tan\theta|^\beta$$

③ 若 $\alpha + \beta < 2$, 则 $\lim_{t \rightarrow 0^+} f(t \cos\theta, t \sin\theta) = t^{\alpha+\beta-2} \frac{|\cos\theta|^\alpha |\sin\theta|^\beta}{\cos^2\theta + t^2 \sin^4\theta} = +\infty$

5.(1) 对于方向 $\mathbf{l} = (\cos\theta, \sin\theta), t > 0$.

$$f(t \cos\theta, t \sin\theta) = \frac{t^4 \sin^2\theta \cos^2\theta}{(t^2 \sin^2\theta + |t \cos\theta|)^3} \leq \frac{t^4 \sin^2\theta \cos^2\theta}{(2\sqrt{t^3 \sin^2\theta} |t \cos\theta|)^3} = \frac{t^4 \sin^2\theta \cos^2\theta}{8(\sin^2\theta |\cos\theta|)^{3/2} \sqrt{t}}$$

$\Rightarrow \lim_{t \rightarrow 0^+} f(t \cos\theta, t \sin\theta) = 0$.

5.(2) $f(x, y) = \frac{x^2 y^2}{(x^2 + |y|)^3}$, 令 $y = x^2$, 则

$$\lim_{\substack{(x, y) \rightarrow (0, 0) \\ y = x^2}} f(x, y) = \lim_{\substack{(x, y) \rightarrow (0, 0) \\ y = x^2}} \frac{x^6}{(x^2 + x^2)^3} = \frac{1}{8} \neq 0. \text{故全极限不存在.} \square$$

$$6.(1) f(x, y) = \frac{x^2 y^2}{x^4 + y^4 - x^2 y^2}$$

因为 x, y 是对称的, 所以 $\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y)$.

$$6.(2) \lim_{\substack{(x, y) \rightarrow (0, 0) \\ y = kx}} f(x, y) = \lim_{\substack{(x, y) \rightarrow (0, 0) \\ y = kx}} \frac{k^2 x^4}{x^4 + k^4 x^4 - k^2 x^4} = \frac{k^2}{1 + k^4 - k^2}. \square$$

求全极限和两个累次极限

$$7.(2) f(x, y) = \frac{\sin(xy)}{x^2 + y^2}$$

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) \text{ 不存在, 因为 } \lim_{\substack{(x, y) \rightarrow (0, 0) \\ y = kx}} \frac{\sin(xy)}{x^2 + y^2} = \lim_{\substack{(x, y) \rightarrow (0, 0) \\ y = kx}} \frac{\sin(kx^2)}{x^2 + k^2 x^2} = \frac{k}{1 + k^2}$$

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y) = \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{\sin(xy)}{x^2 + y^2} = \lim_{x \rightarrow 0} 0 = 0$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{\sin(xy)}{x^2 + y^2} = \lim_{y \rightarrow 0} 0 = 0$$

$$7.(4) f(x, y) = \frac{x^3 + y^2}{x^2 + |y|}$$

$$|f(x, y)| = \left| \frac{x^3 + y^2}{x^2 + |y|} \right| = \left| \frac{x(x^2 + |y|) - x|y| + y^2}{x^2 + |y|} \right| = \left| x + \frac{y^2 - x|y|}{x^2 + |y|} \right| \leq |x| + \left| \frac{y^2 - x|y|}{x^2 + |y|} \right|$$

$$\leq |x| + \left| \frac{y^2 - x|y|}{|y|} \right| = |x| + ||y| - x|| \leq |x| + |y| + x|y|$$

$$\text{于是 } \lim_{(x, y) \rightarrow (0, 0)} |f(x, y)| = \lim_{(x, y) \rightarrow (0, 0)} |x| + |y| + x|y| = 0 \Rightarrow \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 0.$$

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y) = \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{x^3 + y^2}{x^2 + |y|} = \lim_{x \rightarrow 0} x = 0$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{x^3 + y^2}{x^2 + |y|} = \lim_{y \rightarrow 0} |y| = 0$$

$$7.(6) f(x, y) = \frac{x^4 y^4}{(x^2 + y^4)^3}$$

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) \text{ 不存在, 因为 } \lim_{\substack{(x, y) \rightarrow (0, 0) \\ y = \sqrt{kx}}} \frac{x^4 y^4}{(x^2 + y^4)^3} = \lim_{\substack{(x, y) \rightarrow (0, 0) \\ y = \sqrt{kx}}} \frac{k^2 x^6}{((1 + k^2)x^2)^3} = \frac{k^2}{(1 + k^2)^3}$$

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y) = \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{x^4 y^4}{(x^2 + y^4)^3} = \lim_{x \rightarrow 0} 0 = 0$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{x^4 y^4}{(x^2 + y^4)^3} = \lim_{y \rightarrow 0} 0 = 0$$

$$9.(2) \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \frac{\cos(x^3 y^4)}{x^2 + y^2}$$

$$\left| \frac{\cos(x^3 y^4)}{x^2 + y^2} \right| \leq \frac{1}{x^2 + y^2}$$

$$\text{于是 } \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \left| \frac{\cos(x^3 y^4)}{x^2 + y^2} \right| \leq \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \frac{1}{x^2 + y^2} = 0 \Rightarrow \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \frac{\cos(x^3 y^4)}{x^2 + y^2} = 0.$$

$$9.(4) \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow 0}} \left(1 + \frac{1}{x}\right)^{\frac{x^3}{x^2 + y^2}}$$

$$\left(1 + \frac{1}{x}\right)^{\frac{x^3}{x^2 + y^2}} = \exp\left\{\frac{x^3}{x^2 + y^2} \ln\left(1 + \frac{1}{x}\right)\right\} = \exp\left\{\frac{x^3}{x^2 + y^2} \left(\frac{1}{x} + O\left(\frac{1}{x^2}\right)\right)\right\} = \exp\left\{\frac{x^2 + O(x)}{x^2 + y^2}\right\}$$

$$\lim_{\substack{x \rightarrow +\infty \\ y \rightarrow 0}} \frac{x^2 + O(x)}{x^2 + y^2} = \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow 0}} \frac{x^2}{x^2 + y^2} + \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow 0}} \frac{O(x)}{x^2 + y^2}$$

$$\lim_{\substack{x \rightarrow +\infty \\ y \rightarrow 0}} \frac{x^2}{x^2 + y^2} = 1, \quad \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow 0}} \frac{O(x)}{x^2 + y^2} = 0.$$

$$\text{于是 } \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow 0}} \frac{x^2 + O(x)}{x^2 + y^2} = 1$$

$$\lim_{\substack{x \rightarrow +\infty \\ y \rightarrow 0}} \left(1 + \frac{1}{x}\right)^{\frac{x^3}{x^2 + y^2}} = \exp\left\{\lim_{\substack{x \rightarrow +\infty \\ y \rightarrow 0}} \frac{x^2 + O(x)}{x^2 + y^2}\right\} = e.$$

$$9.(5) \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \left(\frac{xy}{x^2 + y^2}\right)^{x^2 + y^2}$$

$$\left(\frac{xy}{x^2 + y^2}\right)^{x^2 + y^2} = \exp\left\{(x^2 + y^2) \ln\left(\frac{xy}{x^2 + y^2}\right)\right\} \leq \exp\left\{(x^2 + y^2) \ln\left(\frac{xy}{2xy}\right)\right\} = \exp\{-(x^2 + y^2) \ln 2\}$$

$$\text{于是 } \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \left(\frac{xy}{x^2 + y^2}\right)^{x^2 + y^2} \leq \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \exp\{-(x^2 + y^2) \ln 2\} = \exp\left\{-\lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} (x^2 + y^2) \ln 2\right\} = 0$$

$$9.(6) \lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2)^{x^2 y^2}$$

$$(x^2 + y^2)^{x^2 y^2} = \exp\{x^2 y^2 \ln(x^2 + y^2)\} = \exp\left\{\frac{x^2 y^2}{x^2 + y^2} (x^2 + y^2) \ln(x^2 + y^2)\right\}$$

$$\text{于是 } \lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2)^{x^2 y^2} = \exp\left\{\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 + y^2} (x^2 + y^2) \ln(x^2 + y^2)\right\}$$

$$= \exp\left\{\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 + y^2} \lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \ln(x^2 + y^2)\right\} = 1$$

(1) $\lim_{\substack{(x,y) \rightarrow (0,0) \\ x \neq y}} \frac{x^3 + y^3}{x - y}$ 不存在

取 $x = y + ky^3$, 则 $\lim_{\substack{(x,y) \rightarrow (0,0) \\ x = y + ky^3}} \frac{x^3 + y^3}{x - y} = \lim_{\substack{(x,y) \rightarrow (0,0) \\ x = y + ky^3}} \frac{(y + ky^3)^3 + y^3}{ky^3} = \frac{2}{k}$

(2) $\lim_{\substack{(x,y) \rightarrow (0,0) \\ x+y \neq 0}} \frac{x^2 y^2}{x^3 + y^3}$

$$\left| \frac{x^2 y^2}{x^3 + y^3} \right| \leq \left| \frac{x^2 y^2}{\sqrt{2}|x^3 y^3|} \right| = \frac{\sqrt{|xy|}}{\sqrt{2}}$$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ x+y \neq 0}} \left| \frac{x^2 y^2}{x^3 + y^3} \right| \leq \lim_{\substack{(x,y) \rightarrow (0,0) \\ x+y \neq 0}} \frac{\sqrt{|xy|}}{\sqrt{2}} = 0$$

$$\Rightarrow \lim_{\substack{(x,y) \rightarrow (0,0) \\ x+y \neq 0}} \frac{x^2 y^2}{x^3 + y^3} = 0.$$

$$1. (1) f(x, y) = \begin{cases} \frac{\sin(x^2 y)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

在 $(x, y) \neq (0, 0)$ 处, $f(x, y)$ 显然连续

只需证 $f(x, y)$ 在 $(0, 0)$ 处连续.

$$\left| \frac{\sin(x^2 y)}{x^2 + y^2} \right| \leq \left| \frac{\sin(x^2 y)}{2xy} \right| = \left| \frac{\sin(x^2 y)}{2x^2 y} \right| \cdot |x|$$

$$\begin{aligned} \text{于是 } \lim_{(x,y) \rightarrow (0,0)} \left| \frac{\sin(x^2 y)}{x^2 + y^2} \right| &\leq \lim_{(x,y) \rightarrow (0,0)} \left| \frac{\sin(x^2 y)}{2x^2 y} \right| \cdot |x| = \lim_{(x,y) \rightarrow (0,0)} \left| \frac{\sin(x^2 y)}{2x^2 y} \right| \cdot \lim_{(x,y) \rightarrow (0,0)} |x| \\ &= \frac{1}{2} \lim_{(x,y) \rightarrow (0,0)} |x| = 0 \end{aligned}$$

于是 $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 y)}{x^2 + y^2} = 0 \Rightarrow f(x, y)$ 在 $(0, 0)$ 处连续.

$$1. (2) f(x, y) = \begin{cases} xy \arctan \frac{1}{x} \arctan \frac{1}{y}, & x \neq 0, y \neq 0 \\ 0, & xy = 0 \end{cases}$$

在 $\{(x, y): xy \neq 0\}$ 处, $f(x, y)$ 显然连续

只需证 $f(x, y)$ 在 $\{(x, y): xy = 0\}$ 处连续.

$$\lim_{\substack{xy \rightarrow 0 \\ x \neq 0, y \neq 0}} \left| xy \arctan \frac{1}{x} \arctan \frac{1}{y} \right| \leq \lim_{\substack{xy \rightarrow 0 \\ x \neq 0, y \neq 0}} \frac{\pi^2}{4} \cdot |xy| = 0$$

于是 $\lim_{\substack{xy \rightarrow 0 \\ x \neq 0, y \neq 0}} xy \arctan \frac{1}{x} \arctan \frac{1}{y} = 0 \Rightarrow f(x, y)$ 在 $\{(x, y): xy = 0\}$ 处连续.

$$2. f(x, y) = \begin{cases} \frac{\sin(xy)}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Pf: 固定 y , $\lim_{x \rightarrow 0} \left| \frac{\sin(xy)}{x^2 + y^2} \right| \leq \lim_{x \rightarrow 0} \left| \frac{\sin(xy)}{y^2} \right| = 0$, 于是 $\lim_{x \rightarrow 0} \frac{\sin(xy)}{x^2 + y^2} = 0$

固定 x , $\lim_{y \rightarrow 0} \left| \frac{\sin(xy)}{x^2 + y^2} \right| \leq \lim_{y \rightarrow 0} \left| \frac{\sin(xy)}{x^2} \right| = 0$, 于是 $\lim_{y \rightarrow 0} \frac{\sin(xy)}{x^2 + y^2} = 0$

但是取 $x = y$, 则 $\lim_{\substack{(x, y) \rightarrow (0, 0) \\ x = y}} \frac{\sin(xy)}{x^2 + y^2} = \lim_{\substack{(x, y) \rightarrow (0, 0) \\ x = y}} \frac{\sin(x^2)}{2x^2} = \frac{1}{2} \neq 0 \Rightarrow f(x, y)$ 不是连续函数. \square

$$3. f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$\forall 0 \leq \theta < 2\pi$, $\lim_{t \rightarrow 0} f(t \cos \theta, t \sin \theta) = \lim_{t \rightarrow 0} \frac{t^3 \cos^2 \theta \sin \theta}{t^4 \cos^4 \theta + t^2 \sin^2 \theta} = \lim_{t \rightarrow 0} \frac{t \cos^2 \theta \sin \theta}{t^2 \cos^4 \theta + \sin^2 \theta}$

若 $\theta = 0$ 或 π , 则 $\frac{t \cos^2 \theta \sin \theta}{t^2 \cos^4 \theta + \sin^2 \theta} = 0 \Rightarrow \lim_{t \rightarrow 0} \frac{t \cos^2 \theta \sin \theta}{t^2 \cos^4 \theta + \sin^2 \theta} = 0$

若 $\theta \neq 0, \pi$, 则 $\lim_{t \rightarrow 0} \left| \frac{t \cos^2 \theta \sin \theta}{t^2 \cos^4 \theta + \sin^2 \theta} \right| \leq \lim_{t \rightarrow 0} \left| \frac{t \cos^2 \theta \sin \theta}{\sin^2 \theta} \right| = \lim_{t \rightarrow 0} \left| \frac{t \cos^2 \theta}{\sin \theta} \right| = 0$

$\Rightarrow \lim_{t \rightarrow 0} \frac{t \cos^2 \theta \sin \theta}{t^2 \cos^4 \theta + \sin^2 \theta} = 0$.

但是 $\lim_{\substack{(x, y) \rightarrow (0, 0) \\ y = x^2}} \frac{x^2 y}{x^4 + y^2} = \lim_{\substack{(x, y) \rightarrow (0, 0) \\ y = x^2}} \frac{x^4}{x^4 + x^4} = \frac{1}{2} \neq 0 \Rightarrow f$ 在 $O(0, 0)$ 不连续. \square

5. (1) 存在 $N > 0$, 使得对于任意 $|x| \geq N$, 都有 $a - 1 \leq f(x) \leq a + 1$.

而连续函数 f 在有界闭集 $\{x \in \mathbb{R}^m : |x| \leq N\}$ 有界.

故 $|f(x)| \leq \max \left\{ \sup_{|x| \leq N} |f(x)|, |a - 1|, |a + 1| \right\}$, $\forall x \in \mathbb{R}^m \Rightarrow f(x)$ 在 \mathbb{R}^m 有界.

5. (2) $\forall \epsilon > 0$, $\exists N > 0$, s.t. $\forall |x| \geq N$, 有 $|f(x) - a| < \frac{\epsilon}{2}$

由于 f 在有界闭集 $\{x \in \mathbb{R}^m : |x| \leq N + 1\}$ 连续, 则 f 在有界闭集 $\{x \in \mathbb{R}^m : |x| \leq N + 1\}$ 一致连续

于是 $\exists 0 < \delta < \frac{1}{2}$, s.t. $\forall |x - y| < \delta$, $|x| \leq N + 1$, $|y| \leq N + 1$, 有 $|f(x) - f(y)| < \epsilon$.

$\forall |x| \geq N$, $|y| \geq N$, 有 $|f(x) - f(y)| \leq |f(x) - a| + |a - f(y)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

于是 $\forall x, y \in \mathbb{R}^m$, $|x - y| < \delta$, 有 $|f(x) - f(y)| < \epsilon$. \square

8. (1) Pf: 取 $\lambda = \frac{1}{|x|}, \forall x \in \mathbb{R}^m - \{0\}$.

则 $f\left(\frac{x}{|x|}\right) = \frac{1}{|x|^\mu} f(x)$, 其中 $\left|\frac{x}{|x|}\right| = 1$.

考虑有界闭集 $\{x \in \mathbb{R}^m : |x| = 1\}$, 由于 f 在 $\mathbb{R}^m - \{0\} \supseteq \{x \in \mathbb{R}^m : |x| = 1\}$ 连续,

故 $f(x)$ 在 $\{x \in \mathbb{R}^m : |x| = 1\}$ 上取到最大值和最小值,

于是 $|f(x)| \leq \max_{|x|=1} |f(x)|, \forall x \in \{x \in \mathbb{R}^m : |x| = 1\}$.

于是 $|f(x)| = \left|f\left(\frac{x}{|x|}\right)\right| \cdot |x|^\mu \leq \max_{|x|=1} |f(x)| \cdot |x|^\mu$. 取 $C = \max_{|x|=1} |f(x)|$ 即可. \square

8. (2) $f(x)$ 在 $\{x \in \mathbb{R}^m : |x| = 1\}$ 上取到最小值 $\min_{|x|=1} f(x) > 0$.

于是 $f(x) = f\left(\frac{x}{|x|}\right) \cdot |x|^\mu \geq \min_{|x|=1} f(x) \cdot |x|^\mu$. 取 $c = \min_{|x|=1} f(x)$ 即可. \square

9. (1) 由于 S^1 是 \mathbb{R}^2 上的有界闭集, 故显然连续的 $f(x, y)$ 在 S^1 上有最大值和最小值.

9. (2) 在 S^1 上取一个点 $(\cos \theta, \sin \theta), \theta \in [0, 2\pi)$ 使得 $f(\cos \theta, \sin \theta) = a$

在 S^1 上取一个点 $(\cos \gamma, \sin \gamma), \gamma \in [0, 2\pi)$ 使得 $f(\cos \gamma, \sin \gamma) = b$

由于 $a < b$, 显然 $\theta \neq \gamma$, 不妨设 $\theta < \gamma$

考虑函数 $\begin{cases} g(t) = f(\cos(\theta + (\gamma - \theta)t), \sin(\theta + (\gamma - \theta)t)) \\ h(t) = f(\cos(\gamma + (2\pi + \theta - \gamma)t), \sin(\gamma + (2\pi + \theta - \gamma)t)) \end{cases}$ 显然连续

$g(0) = h(1) = f(\cos \theta, \sin \theta) = a, g(1) = h(0) = f(\cos \gamma, \sin \gamma) = b$

由连续函数介值性可知: $\forall c \in (a, b)$, 存在 $t_1, t_2 \in (0, 1)$, s.t. $g(t_1) = h(t_2) = c$

而 $(\cos(\theta + (\gamma - \theta)t_1), \sin(\theta + (\gamma - \theta)t_1)) \in \{(\cos x, \sin x) : \theta < x < \gamma\}$

$(\cos(\gamma + (2\pi + \theta - \gamma)t_2), \sin(\gamma + (2\pi + \theta - \gamma)t_2)) \in \{(\cos x, \sin x) : \gamma < x < 2\pi + \theta\}$

$\{(\cos x, \sin x) : \theta < x < \gamma\} \cap \{(\cos x, \sin x) : \gamma < x < 2\pi + \theta\} = \emptyset$.

于是 $(\cos(\theta + (\gamma - \theta)t_1), \sin(\theta + (\gamma - \theta)t_1)) \neq (\cos(\gamma + (2\pi + \theta - \gamma)t_2), \sin(\gamma + (2\pi + \theta - \gamma)t_2))$.

故得证! \square