1. $\sum_{n=1}^{\infty}\left|\frac{a_n}{n}\right|+\sum_{n=1}^{\infty}\left|\frac{b_n}{n}\right| \le \sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2)} \le \sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{|a_0|^2}{2}+\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2)\right]}$

$=\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}}\sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx}<\infty \ldots\ldots (\star)$

故 $\sum_{n=1}^{\infty}\left|\frac{a_n}{n}\right|<\infty, \sum_{n=1}^{\infty}\left|\frac{b_n}{n}\right|<\infty$.故 $\sum_{n=1}^{\infty}\frac{a_n}{n}, \sum_{n=1}^{\infty}\frac{b_n}{n}$ 绝对收敛

(1) 由 $(\star)$ 可知: $\sum_{n=1}^{\infty}\frac{|a_n|+|b_n|}{n} \le \sqrt{\frac{\pi}{3}}\left(\int_{-\pi}^{\pi}|f(x)|^2dx\right)^{\frac{1}{2}}$

(2) 注意到: $\pi - x = \sum_{n=1}^{\infty}\frac{2\sin(nx)}{n}, x\in\mathbb{R}$

只需证: $\sum_{n=1}^{\infty}\frac{b_n}{n} = \frac{1}{2\pi}\int_0^{2\pi}f(x)\sum_{n=1}^{\infty}\frac{2\sin(nx)}{n}dx$

即 $\sum_{n=1}^{\infty}\frac{\frac{1}{\pi}\int_0^{2\pi}f(x)\sin(nx)dx}{n} = \frac{1}{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{f(x)\sin(nx)}{n}dx$

由于 f 在 $[0,2\pi]$ 平方可积(常义 *Riemann*),所以 f 在 $[0,2\pi]$ 有界.

由 *Dirichlet* 判别法:级数 $\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$ 收敛

于是对于任意给定的 $\epsilon>0$,存在 $N>0$,使得对于任意 $n>N$,有 $\left|\sum_{n=N+1}^{\infty}\frac{\sin(nx)}{n}\right|<\frac{\epsilon}{2\sup\limits_{x\in[0,2\pi]}|f(x)|}$

$\left|\sum_{n=1}^{N}\frac{\frac{1}{\pi}\int_0^{2\pi}f(x)\sin(nx)dx}{n}-\frac{1}{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{f(x)\sin(nx)}{n}dx\right|$

$=\left|\frac{1}{\pi}\int_0^{2\pi}f(x)\sum_{n=N+1}^{\infty}\frac{\sin(nx)}{n}dx\right|$

$\le \frac{1}{\pi}\int_0^{2\pi}|f(x)|\left|\sum_{n=N+1}^{\infty}\frac{\sin(nx)}{n}\right|dx$

$\le 2\sup\limits_{x\in[0,2\pi]}|f(x)|\frac{\epsilon}{2\sup\limits_{x\in[0,2\pi]}|f(x)|}=\epsilon$

于是 $\lim\limits_{N\to\infty}\left|\sum_{n=1}^{N}\frac{\frac{1}{\pi}\int_0^{2\pi}f(x)\sin(nx)dx}{n}-\frac{1}{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{f(x)\sin(nx)}{n}dx\right|=0$

于是 $\sum_{n=1}^{\infty}\frac{\frac{1}{\pi}\int_0^{2\pi}f(x)\sin(nx)dx}{n} = \frac{1}{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{f(x)\sin(nx)}{n}dx.\square$

2.由于 f'在$(-\pi,\pi)$平方可积,由 $Parseval$ 恒等式:

$$\sum_{n=1}^{\infty} n^2(|a_n|^2+|b_n|^2)=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f'(x)|^2dx$$

由 $Cauchy$ 不等式:

$$\sum_{n=1}^{\infty} n^2(|a_n|^2+|b_n|^2)\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=1}^{\infty} n^2|a_n|^2\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty} n^2|b_n|^2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\geq\left(\sum_{n=1}^{\infty}(|a_n|+|b_n|)\right)^2$$

于是 $\sum_{n=1}^{\infty}(|a_n|+|b_n|)\leq\sqrt{\sum_{n=1}^{\infty} n^2(|a_n|^2+|b_n|^2)\sum_{n=1}^{\infty}\frac{1}{n^2}}=\sqrt{\frac{\pi}{3}}\left(\int_{-\pi}^{\pi}|f'(x)|^2dx\right)^{\frac{1}{2}}\square$

3.(1) 注意到: $S_n'(x)=\frac{d}{dx}(D_n*f)(x)\overset{\text{卷积求导法则}}{=}(D_n*f')(x)$

归纳可得: $S_n^{(k)}(x)=(D_n*f^{(k)})(x)$

由于 $f^{(m)}$ 在 $(-\pi,\pi)$ 上平方可积, 显然 $f^{(k)}$ 在 $(-\pi,\pi)$ 上平方可积 $(k=0,1,\cdots,m)$

显然 $f^{(k)}$ 在 $(-\pi,\pi)$ 上(绝对)可积 $(k=0,1,\cdots,m)$

于是有 $\lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f^{(k)}(x)-(D_n*f^{(k)})(x)|^2dx=0$

因为 m 是有限的, 所以 $\lim_{n\to\infty}\sum_{k=0}^{m}\int_{-\pi}^{\pi}|f^{(k)}(x)-(D_n*f^{(k)})(x)|^2dx=0$.

(2) 引理: 若圆盘上的可积函数 f 在 θ_0 处可微, 那么 $S_n(\theta_0)\to f(\theta_0)$ (当 $n\to\infty$)

引理证明: 考虑函数 $F(t)=\begin{cases}\frac{f(\theta_0-t)-f(\theta_0)}{t} & \text{若 } t\neq 0\text{, 且 } t<|\pi| \\ -f'(\theta_0) & \text{若 } t=0\end{cases}$, 显然 $F(\theta)$ 在 $(-\pi,\pi)$ 可积

于是 $|S_n(\theta_0)-f(\theta_0)|=\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta_0-t)D_n(t)dt-f(\theta_0)\right|$

$$\overset{\frac{1}{2\pi}\int_{-\pi}^{\pi}D_n(t)dt=1}{=}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta_0-t)D_n(t)dt-\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta_0)D_n(t)dt\right|$$

$$=\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta_0-t)-f(\theta_0)]D_n(t)dt\right|$$

$$=\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}F(t)tD_n(t)dt\right|$$

$$=\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}F(t)\frac{t}{\sin\frac{t}{2}}\sin\left(\left(n+\frac{1}{2}\right)t\right)dt\right|$$

$$=\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}F(t)t\tan\frac{t}{2}\sin(nt)dt+\frac{1}{2\pi}\int_{-\pi}^{\pi}F(t)t\cos(nt)dt\right|$$

其中 $F(t)t\tan\frac{t}{2}$ 在 $(-\pi,\pi)$ 可积, $F(t)t$ 在 $(-\pi,\pi)$ 可积

由 $Riemann-Lebesgue$ 引理:

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}F(t)t\tan\frac{t}{2}\sin(nt)dt=\lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}F(t)t\cos(nt)dt=0$$

于是 $\lim_{n\to\infty}S_n(\theta_0)=f(\theta_0)$, 引理得证!

我们回到原题:

对于 $k\in\{0,1,\cdots,m-1\}$

因为 $f^{(k)}$ 在 $(-\pi,\pi)$ 上可微, 所以我们有 $S_n^{(k)}(x)=f^{(k)}(x),\forall x\in(-\pi,\pi)$

于是由引理可知: $\lim_{n\to\infty}S_n^{(k)}(x)=f^{(k)}(x),\forall x\in(-\pi,\pi)$

结合 $f^{(k)}$ 在 $\mathbb{R}$ 上连续, 于是 $\lim_{n\to\infty}S_n^{(k)}(x)=f^{(k)}(x),\forall x\in[-\pi,\pi]$

于是 $\lim_{n\to\infty}\max_{-\pi\le x\le\pi}|S_n^{(k)}(x)-f^{(k)}(x)|=0,\ \forall k\in\{0,1,\cdots,m-1\}$

由于 m 是有限的, 所以 $\lim_{n\to\infty}\sum_{k=0}^{m-1}\max_{-\pi\le x\le\pi}|S_n^{(k)}(x)-f^{(k)}(x)|=0.\square$

4.(1) 由于$f'(x)$在$(-\pi,\pi)$上平方可积,所以可积.显然$f(x)$在$(-\pi,\pi)$可积

于是由$Parseval$恒等式:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2dx=\frac{|a_0|^2}{2}+\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f'(x)|^2dx=\sum_{n=1}^{\infty}n^2(|a_n|^2+|b_n|^2)$$

于是$\displaystyle\int_{-\pi}^{\pi}|f(x)|^2dx=\frac{\pi}{2}|a_0|^2+\pi\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2)\le\frac{\pi}{2}\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx\right|^2+\pi\sum_{n=1}^{\infty}n^2(|a_n|^2+|b_n|^2)$

$$=\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}f(x)dx\right|^2+\int_{-\pi}^{\pi}|f'(x)|^2dx$$

(2) 若f以π为周期,则

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2dx=\frac{2}{\pi}\int_{0}^{\pi}|f(x)|^2dx$$

考虑2π周期函数$g(x)=f\left(\frac{x}{2}\right)$,$\displaystyle\frac{dg(x)}{dx}=\frac{df\left(\frac{x}{2}\right)}{dx}=\frac{1}{2}f'\left(\frac{x}{2}\right)$

于是$\displaystyle\int_{-\pi}^{\pi}|g(x)|^2dx\le\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}g(x)dx\right|^2+\int_{-\pi}^{\pi}\left|\frac{dg(x)}{dx}\right|^2dx$

即$\displaystyle\int_{-\pi}^{\pi}\left|f\left(\frac{x}{2}\right)\right|^2dx\le\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}f\left(\frac{x}{2}\right)dx\right|^2+\frac{1}{4}\int_{-\pi}^{\pi}\left|f'\left(\frac{x}{2}\right)\right|^2dx$

即$\displaystyle 2\int_{-\pi}^{\pi}\left|f\left(\frac{x}{2}\right)\right|^2d\frac{x}{2}\le\frac{1}{2\pi}\left|2\int_{-\pi}^{\pi}f\left(\frac{x}{2}\right)d\frac{x}{2}\right|^2+\frac{1}{2}\int_{-\pi}^{\pi}\left|f'\left(\frac{x}{2}\right)\right|^2d\frac{x}{2}$

即$\displaystyle 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|f(x)|^2dx\le\frac{1}{2\pi}\left|2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}f(x)dx\right|^2+\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|f'(x)|^2dx$

即$\displaystyle\int_{-\pi}^{\pi}|f(x)|^2dx\le\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}f(x)dx\right|^2+\frac{1}{4}\int_{-\pi}^{\pi}|f'(x)|^2dx.\square$

$$5.(1)\ \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x)-f\left(x-\frac{\pi}{m}\right)\right]e^{-inx}dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx-\frac{1}{2\pi}\int_{-\pi}^{\pi}f\left(x-\frac{\pi}{m}\right)e^{-inx}dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx-\frac{1}{2\pi}\int_{-\pi-\frac{\pi}{m}}^{\pi-\frac{\pi}{m}}f(x)e^{-in\left(x+\frac{\pi}{m}\right)}dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx-e^{-\frac{in\pi}{m}}\frac{1}{2\pi}\int_{-\pi-\frac{\pi}{m}}^{\pi-\frac{\pi}{m}}f(x)e^{-inx}dx$$

$$=\left(1-e^{-\frac{in\pi}{m}}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx$$

$$=\left(1-\cos\frac{nx}{m}+i\sin\frac{nx}{m}\right)\frac{a_n-ib_n}{2}$$

而 $\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x)-f\left(x-\frac{\pi}{m}\right)\right]e^{-inx}dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x)-f\left(x-\frac{\pi}{m}\right)\right]\left(\cos\frac{nx}{m}-i\sin\frac{nx}{m}\right)dx$

显然,比较实部和虚部就有:

$$\begin{cases} a_n\left(1-\cos\frac{n\pi}{m}\right)+b_n\sin\frac{n\pi}{m}=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(x)-f\left(x-\frac{\pi}{m}\right)\right]\cos nx\,dx, \\ -a_n\sin\frac{n\pi}{m}+b_n\left(1-\cos\frac{n\pi}{m}\right)=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(x)-f\left(x-\frac{\pi}{m}\right)\right]\sin nx\,dx; \end{cases}$$

5.(2) 记 $g_m(x)=f(x)-f\left(x-\frac{\pi}{m}\right)$

由(1)可知: $\widehat{g_m}(n)=\left(1-e^{-\frac{in\pi}{m}}\right)\hat{f}(n)$

由 $Parseval$ 恒等式: $\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(x)|^2dx=\sum_{n=-\infty}^{+\infty}\left|\widehat{g_m}(n)\right|^2\geq\sum_{2^k+1\leq|n|\leq 2^{k+1}}\left|\widehat{g_m}(n)\right|^2$

令 $m=2^{k+2}$, 则 $\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(x)|^2dx\geq\sum_{2^k+1\leq|n|\leq 2^{k+1}}\left|\widehat{g_{2^{k+2}}}(n)\right|^2=\sum_{2^k+1\leq|n|\leq 2^{k+1}}\left|1-e^{-\frac{in\pi}{2^{k+2}}}\right|^2\left|\hat{f}(n)\right|^2$

$$\geq\sum_{2^k+1\leq|n|\leq 2^{k+1}}\left|\frac{in\pi}{2^{k+2}}\right|^2\left|\hat{f}(n)\right|^2\geq\frac{\pi^2}{16}\sum_{2^k+1\leq|n|\leq 2^{k+1}}\left|\hat{f}(n)\right|^2$$

由于 $\sum_{2^k+1\leq n\leq 2^{k+1}}\left(|a_n|^2+|b_n|^2\right)=\sum_{2^k+1\leq n\leq 2^{k+1}}\left(\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx\right|^2+\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx\right|^2\right)$

$$=\sum_{2^k+1\leq n\leq 2^{k+1}}\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)e^{inx}dx\right|^2=\sum_{2^k+1\leq n\leq 2^{k+1}}\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx\right|^2$$

$$=\frac{1}{2}\left[\sum_{2^k+1\leq n\leq 2^{k+1}}\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)e^{inx}dx\right|^2+\sum_{2^k+1\leq n\leq 2^{k+1}}\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx\right|^2\right]$$

$$=2\sum_{2^k+1\leq n\leq 2^{k+1}}\left[\left|\hat{f}(n)\right|^2+\left|\hat{f}(-n)\right|^2\right]=2\sum_{2^k+1\leq|n|\leq 2^{k+1}}\left|\hat{f}(n)\right|^2$$

$\leq 2\frac{1}{\frac{\pi^2}{16}}\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(x)|^2dx=\frac{\frac{16}{\pi^2}}{\pi}\int_{-\pi}^{\pi}\left|f(x)-f\left(x-\frac{\pi}{2^{k+2}}\right)\right|^2dx$, 取 $C=\frac{16}{\pi^2}$ 即可.

5.(3) 又因为 f 在 $[-\pi,\pi]$ 上 α 赫尔德连续 $\left(\alpha>\frac{1}{2}\right)$, 于是

对于任意 $\forall x,y\in[-\pi,\pi]$, 存在 $K>0$, 使得 $|f(x)-f(y)|\le K|x-y|^{\alpha}$

显然有 $\forall x,y\in\mathbb{R}$, 存在 $K>0$, 使得 $|f(x)-f(y)|\le K|x-y|^{\alpha}$ (由凸性)

$$\sum_{n\ge1}(|a_n|+|b_n|)=\sum_{k\ge1}\sum_{2^k+1\le n\le 2^{k+1}}(|a_n|+|b_n|)\le\sum_{k\ge1}\sqrt{\sum_{2^k+1\le n\le 2^{k+1}}(1+1)\sum_{2^k+1\le n\le 2^{k+1}}(|a_n|^2+|b_n|^2)}$$

$$\le\sum_{k\ge1}\sqrt{2^{k+1}\sum_{2^k+1\le n\le 2^{k+1}}(|a_n|^2+|b_n|^2)}\le\sum_{k\ge1}\sqrt{2^{k+1}\frac{16}{\pi^3}\int_{-\pi}^{\pi}\left|f(x)-f\left(x-\frac{\pi}{2^{k+2}}\right)\right|^2dx}$$

$$\le\sum_{k\ge1}\sqrt{2^{k+1}\frac{16}{\pi^3}K^2\int_{-\pi}^{\pi}\left|\frac{\pi}{2^{k+2}}\right|^{2\alpha}dx}=K\sum_{k\ge1}\sqrt{2^{k+1}\frac{16}{\pi^3}\left|\frac{\pi}{2^{k+2}}\right|^{2\alpha}(2\pi)}=K\sum_{k\ge1}\sqrt{2^{k+2}\frac{16}{\pi^2}\left|\frac{\pi}{2^{k+2}}\right|^{2\alpha}}$$

$$=K4\pi^{\frac{\alpha-1}{2}}\sum_{k\ge1}\left(\frac{1}{2^{2\alpha-1}}\right)^{(k+2)}=4\pi^{\frac{\alpha-1}{2}}\frac{2^{3-4\alpha}}{2^{2\alpha}-2}K$$

其中 $K\ge\frac{|f(x)-f(y)|}{|x-y|^{\alpha}}$, 故 $K\ge\sup\limits_{-\infty<y<x<+\infty}\frac{|f(x)-f(y)|}{|x-y|^{\alpha}}$, 我们直接取 $K=\sup\limits_{-\infty<y<x<+\infty}\frac{|f(x)-f(y)|}{|x-y|^{\alpha}}$

就有 $\sum\limits_{n\ge1}(|a_n|+|b_n|)\le\pi^{\frac{\alpha-1}{2}}\frac{2^{4-4\alpha}}{2^{2\alpha-1}-1}\cdot\sup\limits_{-\infty<y<x<+\infty}\frac{|f(x)-f(y)|}{|x-y|^{\alpha}}$.□

6.①由 *Dirichlet* 判别法可知 $f(x)=\sum\limits_{n=1}^{\infty}\frac{\sin nx}{n^p}$ 收敛

②对于任意不包含 $2m\pi,m\in\mathbb{Z}$ 的闭区间 $[a,b]$, 我们有 $\{\sin nx\}_{n\ge1}$ 的部分和关于 x 一致有界, $\frac{1}{n^p}$ 递减趋于0

于是 $f(x)=\sum\limits_{n=1}^{\infty}\frac{\sin nx}{n^p}$ 在 $[a,b]$ 上一致收敛, 也就是在 $\mathbb{R}-\{2m\pi:m\in\mathbb{Z}\}$ 内闭一致收敛.

由 *Abel* 连续性定理: $f(x)$ 是 $[a,b]$ 上连续函数的和, 所以 $f(x)$ 也在 $[a,b]$ 上连续.

由 $[a,b]$ 任意性可知: $f(x)$ 在 $\mathbb{R}-\{2m\pi:m\in\mathbb{Z}\}$ 上连续.

③若 $f(x)=\sum\limits_{n=1}^{\infty}\frac{\sin nx}{n^p}\left(0<p<\frac{1}{2}\right)$ 在 $[0,2\pi]$ 平方可积, 则 $f(x)$ 在在 $[0,2\pi]$ 黎曼可积

那么由 *Parseval* 恒等式: $\sum\limits_{n=1}^{\infty}\left|\frac{1}{n^p}\right|^2=\frac{1}{\pi}\int_0^{2\pi}|f(x)|^2dx<\infty$, 但是这与 $\sum\limits_{n=1}^{\infty}\frac{1}{n^{2p}}\ (0<2p<1)$ 发散矛盾!

故 $f(x)=\sum\limits_{n=1}^{\infty}\frac{\sin nx}{n^p}\left(0<p<\frac{1}{2}\right)$ 在 $[0,2\pi]$ 并不平方可积□

$f(x)$ 更不可能黎曼可积了, 因为函数黎曼可积(常义)说明函数在有限区间上积分, 且函数值有限,

于是黎曼可积蕴含平方可积.□

④出于兴趣,我们证明更强的:$p\in(0,1)$时,$f(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n^p}$也不会在$[0,2\pi]$黎曼可积.

若$f(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n^p}$ $(0<p<1)$在$[0,2\pi]$黎曼可积(常义),故$f(x)$在$[0,2\pi]$有界.将$f(x)$进行2π周期延拓.

考虑共轭Dirichlet核(conjugate Dirichlet kernel):$\widetilde{D_N}(x)=\sum_{|n|\le N}\operatorname{sgn}(n)e^{inx}$,这里$\operatorname{sgn}(n)=\begin{cases}1 & 当n>0\\ 0 & 当n=0\\ -1 & 当n<0\end{cases}$.

Check1: $\widetilde{D_N}(x)=i\frac{\cos\frac{x}{2}-\cos\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}}$

记$w=e^{ix}$,则$\widetilde{D_N}(x)=\sum_{-N\le n\le -1}-w^{-n}+\sum_{1\le n\le N}w^n=-\frac{w^{-1}-w^{-N-1}}{1-w^{-1}}+\frac{w-w^{N+1}}{1-w}=\frac{w^{\frac{1}{2}}+w^{-\frac{1}{2}}-w^{N+\frac{1}{2}}-w^{-N-\frac{1}{2}}}{w^{-\frac{1}{2}}-w^{\frac{1}{2}}}$

$=i\frac{\cos\frac{x}{2}-\cos\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}}=\frac{2\sin\left(\frac{N+1}{2}x\right)\sin\left(\frac{Nx}{2}\right)}{\sin\frac{x}{2}}i$

Check2: $\int_{-\pi}^{\pi}\left|\widetilde{D_N}(x)\right|dx\le c\ln N$,对于$N\ge 2$

我们有$\int_{-\pi}^{\pi}\left|\widetilde{D_N}(x)\right|dx=\int_{-\pi}^{\pi}\left|\frac{2\sin\left(\frac{N+1}{2}x\right)\sin\left(\frac{Nx}{2}\right)}{\sin\frac{x}{2}}\right|dx\le 4\int_{-\pi}^{\pi}\left|\frac{\sin\left(\frac{Nx}{2}\right)}{x}\right|dx=8\int_0^{\pi}\frac{\left|\sin\left(\frac{Nx}{2}\right)\right|}{x}dx$

$=8\int_{\frac{\pi}{2}}^{\frac{N\pi}{2}}\frac{|\sin(x)|}{x}dx+8\int_0^{\frac{\pi}{2}}\frac{\sin x}{x}dx\le 8\int_{\frac{\pi}{2}}^{\frac{N\pi}{2}}\frac{1}{x}dx+c_1\le c\ln N$

Check3:若f黎曼可积,则$\left(f*\widetilde{D_N}\right)(0)=O(\ln N)$

由于f黎曼可积,所以f有界

于是$\left(f*\widetilde{D_N}\right)(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\widetilde{D_N}(0-x)dx\le\sup_{x\in[-\pi,\pi]}|f(x)|\cdot c\ln N$

于是$\left(f*\widetilde{D_N}\right)(0)=O(\ln N)$

Check4:事实上,$\left(f*\widetilde{D_N}\right)(0)=\sum_{1\le n\le N}\frac{1}{n^p}=O(N^{1-p})$

由于f黎曼可积,则$\hat{f}(n)=\frac{1}{n^p}$,$\hat{f}(-n)=-\frac{1}{n^p}$ $(n>0)$,$\hat{f}(0)=0$

$\left(f*\widetilde{D_N}\right)(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\widetilde{D_N}(0-x)dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\sum_{|n|\le N}\operatorname{sgn}(n)e^{-inx}dx=\sum_{|n|\le N}\operatorname{sgn}(n)\hat{f}(n)$

$=2\sum_{n\ge 1}\frac{1}{n^p}=O(N^{1-p})$

而$\ln(N)=O(N^{1-p})$ $(N\to\infty)$矛盾!

于是$f(x)$在$[0,2\pi]$并不黎曼可积.□

$1.f(x)=\sum_{i=1}^{\infty}a_i\varphi_i(x)$满足$\lim_{N\to\infty}\int_a^b\left|f(x)-\sum_{i=1}^{N}a_i\varphi_i(x)\right|^2dx=0$

由Cauchy不等式:

$$0\le\int_a^b\left|f(x)w(x)-\sum_{i=1}^{N}a_i\varphi_i(x)w(x)\right|dx\le\left[\int_a^b\left|f(x)-\sum_{i=1}^{N}a_i\varphi_i(x)\right|^2dx\int_a^b|w(x)|^2dx\right]^{\frac{1}{2}}\to0(N\to\infty)$$

于是$\lim_{N\to\infty}\int_a^b\left|f(x)w(x)-\sum_{i=1}^{N}a_i\varphi_i(x)w(x)\right|dx=0$

于是$\lim_{N\to\infty}\int_a^b\left|\sum_{i=N+1}^{\infty}a_i\varphi_i(x)w(x)\right|dx=0$

对于任意给定的$m\ge1$,我们有$\left|\lim_{N\to\infty}\int_a^b\left(\sum_{i=N+1}^{\infty}a_i\varphi_i(x)w(x)\right)\varphi_m(x)dx\right|$

$$\le\lim_{N\to\infty}\sup_{x\in[a,b]}|\varphi_m(x)|\int_a^b\left|\sum_{i=N+1}^{\infty}a_i\varphi_i(x)w(x)\right|dx=0$$

断言:$\int_a^b\sum_{i=1}^{\infty}a_i\varphi_i(x)\varphi_m(x)w(x)dx=\sum_{i=1}^{\infty}\int_a^b a_i\varphi_i(x)\varphi_m(x)w(x)dx$

$$\lim_{N\to\infty}\left|\int_a^b\sum_{i=1}^{\infty}a_i\varphi_i(x)\varphi_m(x)w(x)dx-\sum_{i=1}^{N}\int_a^b a_i\varphi_i(x)\varphi_m(x)w(x)dx\right|$$

$$=\lim_{N\to\infty}\left|\int_a^b\sum_{i=N+1}^{\infty}a_i\varphi_i(x)\varphi_m(x)w(x)dx\right|=0$$

于是$\int_a^b f(x)\varphi_m(x)w(x)dx=\int_a^b\sum_{i=1}^{\infty}a_i\varphi_i(x)\varphi_m(x)w(x)dx=\sum_{i=1}^{\infty}\int_a^b a_i\varphi_i(x)\varphi_m(x)w(x)dx=a_m\int_a^b\varphi_m^2(x)w(x)dx$

$\Rightarrow a_m=\dfrac{\int_a^b f(x)\varphi_m(x)w(x)dx}{\int_a^b\varphi_m^2(x)w(x)dx}$,$\forall m\ge1$.被唯一确定!□

$2.f(x)=x-[x]$

$$\Rightarrow\hat{f}(n)=\int_0^1 f(x)e^{-2\pi inx}dx=\int_0^1(x-[x])e^{-2\pi inx}dx=\int_0^1 xe^{-2\pi inx}dx=\frac{1}{-2\pi in}\int_0^1 xde^{-2\pi inx}$$

$$=\frac{1}{-2\pi in}\left(1-\int_0^1 e^{-2\pi inx}dx\right)=\frac{1}{-2\pi in}+\frac{1}{2\pi in}\int_0^1 e^{-2\pi inx}dx=\frac{1}{-2\pi in},n\ne0$$

$$\hat{f}(0)=\int_0^1 f(x)dx=\int_0^1(x-[x])dx=\int_0^1 xdx=\frac{1}{2}$$

$$于是f(x)=\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{2\pi in}=\frac{1}{2}+\sum_{|n|\ge1}\hat{f}(n)e^{2\pi in}=\frac{1}{2}-\frac{1}{2\pi i}\sum_{|n|\ge1}\frac{e^{2\pi in}}{n}=\frac{1}{2}-\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{e^{2\pi in}-e^{-2\pi in}}{n}$$

$$=\frac{1}{2}-\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{2i\sin(2\pi n)}{n}=\frac{1}{2}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi n)}{n}$$

$4.f(x)=x$ 在 $[0,l]$ 上展开

$$(1)\ \hat{f}(n)=\frac{1}{l}\int_0^l xe^{-in\frac{2\pi}{l}x}dx=\frac{1}{l}\frac{1}{-in\frac{2\pi}{l}}\int_0^l xde^{-in\frac{2\pi}{l}x}=\frac{1}{-2\pi in}\left(l-\int_0^l e^{in\frac{2\pi}{l}x}dx\right)=-\frac{l}{2\pi in},n\neq 0$$

$$\hat{f}(0)=\frac{1}{l}\int_0^l xdx=\frac{l}{2}$$

$$f(x)\sim\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{in\frac{2\pi}{l}x}=\frac{l}{2}+\sum_{n=1}^{\infty}-\frac{l}{2\pi in}\left(e^{in\frac{2\pi}{l}x}-e^{-in\frac{2\pi}{l}x}\right)=\frac{l}{2}+\sum_{n=1}^{\infty}-\frac{l}{2\pi in}\left(2i\sin\left(\frac{2\pi nx}{l}\right)\right)$$

$$=\frac{l}{2}-\sum_{n=1}^{\infty}\frac{l\sin\left(\frac{2\pi nx}{l}\right)}{\pi n}$$

$$\frac{l}{2}-\sum_{n=1}^{\infty}\frac{l\sin\left(\frac{2\pi nx}{l}\right)}{\pi n}=\begin{cases}\frac{l}{2},\text{当 }x=0,l\\ x,\text{当 }0<x<l\end{cases}$$

$(2)\ f(x)=x$ 展开成余弦级数 $\left\{\cos\frac{n\pi x}{l}\right\}_{n\geq 0}$

对 $f(x)$ 进行偶延拓, $F(x):=\begin{cases}x,\text{当 }0\leq x\leq l\\ -x,\text{当 }-l<x<0\end{cases}$

于是 $\hat{F}(n)=\frac{1}{2l}\int_{-l}^{l}F(x)e^{-inx\frac{\pi}{l}}dx=\frac{1}{2l}\int_{-l}^{0}F(x)e^{-inx\frac{\pi}{l}}dx+\frac{1}{2l}\int_0^l F(x)e^{-inx\frac{\pi}{l}}dx$

$$=\frac{1}{2l}\int_0^l F(-x)e^{inx\frac{\pi}{l}}dx+\frac{1}{2l}\int_0^l F(x)e^{-inx\frac{\pi}{l}}dx$$

$$=\frac{1}{2l}\left[\int_0^l F(-x)e^{inx\frac{\pi}{l}}dx+\int_0^l F(x)e^{-inx\frac{\pi}{l}}dx\right]$$

$$=\frac{1}{2l}\left[\int_0^l F(x)\left(e^{inx\frac{\pi}{l}}+e^{-inx\frac{\pi}{l}}\right)dx\right]$$

$$=\frac{1}{l}\int_0^l F(x)\left(\cos\frac{nx\pi}{l}\right)dx$$

$$=\frac{1}{l}\int_0^l x\left(\cos\frac{nx\pi}{l}\right)dx$$

$$=\frac{l(1-(-1)^n)}{n^2\pi^2},n\neq 0$$

$$\hat{F}(0)=\frac{1}{2l}\int_{-l}^{l}F(x)dx=\frac{1}{l}\int_0^l F(x)dx=\frac{l}{2}$$

$$F(x)\sim\sum_{n=-\infty}^{\infty}\hat{F}(n)e^{inx\frac{\pi}{l}}=\frac{l}{2}+\sum_{n\neq 0}\hat{F}(n)e^{inx\frac{\pi}{l}}=\frac{l}{2}+\sum_{n\neq 0}\frac{l(1-(-1)^n)}{n^2\pi^2}e^{inx\frac{\pi}{l}}=\frac{l}{2}+\frac{4l}{\pi^2}\sum_{n\text{ odd }\geq 1}\frac{\cos\frac{n\pi x}{l}}{n^2}$$

于是 $x=\frac{l}{2}+\frac{4l}{\pi^2}\sum_{n\text{ odd }\geq 1}\frac{\cos\frac{n\pi x}{l}}{n^2},\forall x\in[0,l]$

(3) $f(x)=x$, 考虑展开成正弦级数 $\left\{\sin\frac{n\pi x}{l}\right\}_{n\geq 1}$

奇延拓 $F(x)=\begin{cases} x,\ if\ 0<x\leq l \\ 0, if\ x=0 \\ x,\ if\ -l<x<0 \end{cases}$

于是 $\widehat{F}(n)=\frac{1}{2l}\int_{-l}^{l}F(x)e^{-inx\frac{\pi}{l}}dx=\frac{1}{2l}\int_{-l}^{l}xe^{-inx\frac{\pi}{l}}dx=\frac{1}{2l}\int_{-l}^{0}xe^{-inx\frac{\pi}{l}}dx+\frac{1}{2l}\int_{0}^{l}xe^{-inx\frac{\pi}{l}}dx$

$=\frac{1}{2l}\int_{0}^{l}-xe^{inx\frac{\pi}{l}}dx+\frac{1}{2l}\int_{0}^{l}xe^{-inx\frac{\pi}{l}}dx=\frac{1}{2l}\int_{0}^{l}x\left(e^{-inx\frac{\pi}{l}}-e^{inx\frac{\pi}{l}}\right)dx$

$=-\frac{i}{l}\int_{0}^{l}x\sin\frac{n\pi x}{l}dx=-\frac{i}{l}\int_{0}^{l}x\sin\frac{n\pi x}{l}dx=\frac{(-1)^{n}i}{\pi nl}$

于是 $F(x)\sim\sum_{n=-\infty}^{\infty}\widehat{F}(n)e^{inx\frac{\pi}{l}}=\sum_{n\neq 0}\frac{(-1)^{n}i}{\pi nl}e^{inx\frac{\pi}{l}}=\sum_{n\geq 1}\left(\frac{(-1)^{n}i}{\pi nl}e^{inx\frac{\pi}{l}}-\frac{(-1)^{-n}i}{\pi nl}e^{-inx\frac{\pi}{l}}\right)$

$=\sum_{n\geq 1}\left(\frac{(-1)^{n}i}{\pi nl}\cdot 2i\sin\frac{n\pi x}{l}\right)=\sum_{n\geq 1}\frac{(-1)^{n+1}\sin\frac{n\pi x}{l}}{\pi nl}$

$\sum_{n\geq 1}\frac{(-1)^{n+1}\sin\frac{n\pi x}{l}}{\pi nl}=\begin{cases} x, if\ x\in[0,l) \\ 0,\ if\ x=l \end{cases}$

(4) $f(x)=x, x\in[0,l]$, 按照 $\left\{\sin\frac{n\pi}{2l}\right\}_{n \text{ odd } \geq 1}$

延拓到 $[-2l,2l]$, $F(x)=\begin{cases}-2l-x, \ if \ -2l\leq x<-l\\ x, \ if \ -l\leq x\leq l\\ 2l-x, \ if \ l<x\leq 2l\end{cases}$

于是 $\widehat{F}(n)=\frac{1}{4l}\int_{-2l}^{2l}F(x)e^{-inx\frac{2\pi}{4l}}dx=\frac{1}{4l}\int_{-2l}^{2l}F(x)e^{-\frac{in\pi x}{2l}}dx$

$=\frac{1}{4l}\int_{-2l}^{0}F(x)e^{-\frac{in\pi x}{2l}}dx+\frac{1}{4l}\int_{0}^{2l}F(x)e^{-\frac{in\pi x}{2l}}dx=\frac{1}{4l}\int_{0}^{2l}F(-x)e^{\frac{in\pi x}{2l}}dx+\frac{1}{4l}\int_{0}^{2l}F(x)e^{-\frac{in\pi x}{2l}}dx$

$=-\frac{1}{4l}\int_{0}^{2l}F(x)\left(e^{\frac{in\pi x}{2l}}-e^{-\frac{in\pi x}{2l}}\right)dx=-\frac{1}{4l}\int_{0}^{2l}F(x)\left(2i\sin\frac{n\pi x}{2l}\right)dx=-\frac{i}{2l}\int_{0}^{2l}F(x)\left(\sin\frac{n\pi x}{2l}\right)dx$

$=-\frac{i}{2l}\int_{0}^{l}F(x)\left(\sin\frac{n\pi x}{2l}\right)dx-\frac{i}{2l}\int_{l}^{2l}F(x)\left(\sin\frac{n\pi x}{2l}\right)dx$

$=-\frac{i}{2l}\int_{0}^{l}F(x)\left(\sin\frac{n\pi x}{2l}\right)dx-\frac{i}{2l}\int_{0}^{l}F(2l-x)\left(\sin\frac{n\pi(2l-x)}{2l}\right)dx$

$=-\frac{i}{2l}\int_{0}^{l}F(x)\left(\sin\frac{n\pi x}{2l}\right)dx\,[1-(-1)^n]$

$=-\frac{i}{2l}\int_{0}^{l}F(x)\left(\sin\frac{n\pi x}{2l}\right)dx\cdot 2\chi_{\{odd\ number\}}(n)$

$=-\frac{i}{2l}\int_{0}^{l}x\left(\sin\frac{n\pi x}{2l}\right)dx\cdot 2\chi_{\{odd\ number\}}(n)$

$=\frac{i}{2l}\int_{0}^{l}xd\left(\cos\frac{n\pi x}{2l}\right)\cdot\frac{2l}{n\pi}\cdot 2\chi_{\{odd\ number\}}(n)$

$=\frac{i}{2l}\frac{2l}{n\pi}\left[\left(x\cos\frac{n\pi x}{2l}\right)\Big|_{0}^{l}-\int_{0}^{l}\left(\cos\frac{n\pi x}{2l}\right)dx\right]\cdot 2\chi_{\{odd\ number\}}(n)$

$=\frac{i}{2l}\frac{2l}{n\pi}\left[\left(l\cos\frac{n\pi}{2}\right)-\int_{0}^{l}\left(\cos\frac{n\pi x}{2l}\right)dx\right]\cdot 2\chi_{\{odd\ number\}}(n)$

$=-\frac{i}{n\pi}\int_{0}^{l}\left(\cos\frac{n\pi x}{2l}\right)dx\cdot 2\chi_{\{odd\ number\}}(n)$

$=-\frac{i}{n\pi}\int_{0}^{l}d\left(\sin\frac{n\pi x}{2l}\right)\cdot\frac{2l}{n\pi}\cdot 2\chi_{\{odd\ number\}}(n)$

$=-\frac{2il}{n^2\pi^2}\int_{0}^{l}d\left(\sin\frac{n\pi x}{2l}\right)\cdot 2\chi_{\{odd\ number\}}(n)$

$=-\frac{2il}{n^2\pi^2}\left(\sin\frac{n\pi x}{2l}\right)\Big|_{0}^{l}\cdot 2\chi_{\{odd\ number\}}(n)$

$=-\frac{2il}{n^2\pi^2}\left(\sin\frac{n\pi l}{2l}\right)\cdot 2\chi_{\{odd\ number\}}(n)$

$=-\frac{2il}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\cdot 2\chi_{\{odd\ number\}}(n)$

$=\frac{2il}{n^2\pi^2}\cdot(-1)^{\frac{n+1}{2}}\cdot 2\chi_{\{odd\ number\}}(n), n\neq 0$

$\hat{f}(0)=0$

于是 $F(x)\sim \sum_{|n|\neq 0} \frac{2il}{n^2\pi^2}\cdot(-1)^{\frac{n+1}{2}}\cdot 2\chi_{\{odd\ number\}}(n)\cdot e^{\frac{in\pi x}{2l}}$

$= \sum_{n\geq 1}\left[\frac{2il}{n^2\pi^2}\cdot(-1)^{\frac{n+1}{2}}\cdot 2\chi_{\{odd\ number\}}(n)\cdot e^{\frac{in\pi x}{2l}} + \frac{2il}{n^2\pi^2}\cdot(-1)^{\frac{-n+1}{2}}\cdot 2\chi_{\{odd\ number\}}(-n)\cdot e^{-\frac{in\pi x}{2l}}\right]$

$= \sum_{n\geq 1}\left[\frac{2il}{n^2\pi^2}\cdot(-1)^{\frac{n+1}{2}}\cdot 2\chi_{\{odd\ number\}}(n)\cdot e^{\frac{in\pi x}{2l}} - \frac{2il}{n^2\pi^2}\cdot(-1)^{\frac{n+1}{2}}\cdot 2\chi_{\{odd\ number\}}(n)\cdot e^{-\frac{in\pi x}{2l}}\right]$

$= \sum_{n\geq 1}\left[\frac{2il}{n^2\pi^2}\cdot(-1)^{\frac{n+1}{2}}\cdot 2\chi_{\{odd\ number\}}(n)\cdot\left(e^{\frac{in\pi x}{2l}} - e^{-\frac{in\pi x}{2l}}\right)\right]$

$= \sum_{n\geq 1}\left[\frac{2il}{n^2\pi^2}\cdot(-1)^{\frac{n+1}{2}}\cdot 2\chi_{\{odd\ number\}}(n)\cdot\left(2i\sin\frac{n\pi x}{2l}\right)\right]$

$= \sum_{n\geq 1}\left[\frac{4l}{n^2\pi^2}\cdot(-1)^{\frac{n-1}{2}}\cdot 2\chi_{\{odd\ number\}}(n)\cdot\left(\sin\frac{n\pi x}{2l}\right)\right]$

$= \sum_{n\ \text{odd}\ \geq 1}\left[\frac{8l}{n^2\pi^2}\cdot(-1)^{\frac{n-1}{2}}\cdot\left(\sin\frac{n\pi x}{2l}\right)\right]$

$x = \sum_{n\ \text{odd}\ \geq 1}\left[\frac{8l}{n^2\pi^2}\cdot(-1)^{\frac{n-1}{2}}\cdot\left(\sin\frac{n\pi x}{2l}\right)\right], \forall x\in[0,l]$

4.(5) $f(x)=x, x\in[0,l]$,按函数系 $\left\{\cos\frac{n\pi x}{2l}\right\}_{n\ \text{odd}\ \geq 1}$ 展开

n 为奇数时,

$\hat{f}\left(\frac{n}{4}\right)=\frac{1}{l}\int_0^l f(x)e^{-i\frac{n\pi x}{2l}}dx = \frac{2}{n\pi}\int_0^l x d\sin\frac{n\pi x}{2l} = \frac{2}{n\pi}\left(l\sin\frac{n\pi}{2} - \int_0^l \sin\frac{n\pi x}{2l}dx\right) = 2\cdot\left(\frac{l\cdot(-1)^{\frac{n-1}{2}}}{n\pi} - \frac{2l}{n^2\pi^2}\right)$

于是 $f(x)\sim \sum_{n=-\infty}^{\infty}\hat{f}\left(\frac{n}{4}\right)e^{inx\frac{\pi}{2l}} = \sum_{n=-\infty}^{\infty} 2\cdot\left(\frac{l\cdot(-1)^{\frac{n-1}{2}}}{n\pi} - \frac{2l}{n^2\pi^2}\right)e^{inx\frac{\pi}{2l}} = 4\sum_{n\ \text{odd}\ \geq 1}\left[\frac{l}{n\pi}\cdot(-1)^{\frac{n-1}{2}} - \frac{2l}{n^2\pi^2}\right]\cos\frac{n\pi x}{2l}$

于是 $\sum_{n\ \text{odd}\ \geq 1}\left[\frac{4l}{n\pi}\cdot(-1)^{\frac{n-1}{2}} - \frac{8l}{n^2\pi^2}\right]\cos\frac{n\pi x}{2l} = \begin{cases} x, if\ x\in[0,l) \\ 0, if\ x=l \end{cases}$

5.(1) 应用傅里叶展开求微分方程边值问题的解

$$\begin{cases} y''(x)+y(x)=\sin^3\pi x-\sin\pi x,\ 0<x<1 \\ y(0)=y(1)=0; \end{cases}$$

考虑到边界条件,我们将待求的解进行正弦展开:

$$y(x)=\sum_{n=1}^{\infty}A_n\sin(n\pi x),y''(x)=\sum_{n=1}^{\infty}(-n^2\pi^2A_n)\sin(n\pi x)$$

于是, $y''(x)+y(x)=\sum_{n=1}^{\infty}A_n\sin(n\pi x)+\sum_{n=1}^{\infty}(-n^2\pi^2A_n)\sin(n\pi x)=\sum_{n=1}^{\infty}(1-n^2\pi^2)A_n\sin(n\pi x)$

$$\sin^3\pi x-\sin\pi x=-\frac{\sin\pi x+\sin(3\pi x)}{4}$$

于是 $\begin{cases} (1-\pi^2)A_1=-\frac{1}{4} \\ (1-9\pi^2)A_3=-\frac{1}{4} \\ A_n=0,\ if\ n\neq 1,3 \end{cases} \Rightarrow y(x)=\frac{1}{4}\left[\frac{\sin(\pi x)}{\pi^2-1}+\frac{\sin(3\pi x)}{9\pi^2-1}\right].\square$

5.(3) 应用傅里叶展开求微分方程边值问题的解

$$\begin{cases} y''(x)+2\pi y(x)=2\sin\frac{\pi x}{2},\ 0<x<1 \\ y(0)=y'(1)=0; \end{cases}$$

考虑到边界条件,我们将待求的解按 $\left\{\sin\frac{n\pi x}{2}\right\}_{n\ \mathrm{odd}\ \geq 1}$ 展开:

设 $y(x)=\sum_{n\ \mathrm{odd}\ \geq 1}A_n\sin\frac{n\pi x}{2},y''(x)=\sum_{n\ \mathrm{odd}\ \geq 1}\left(-\frac{n^2\pi^2A_n}{4}\right)\sin\frac{n\pi x}{2}$

于是, $2\sin\frac{\pi x}{2}=y''(x)+2\pi y(x)=\sum_{n\ \mathrm{odd}\ \geq 1}\left(-\frac{n^2\pi^2A_n}{4}\right)\sin\frac{n\pi x}{2}+2\pi\sum_{n\ \mathrm{odd}\ \geq 1}A_n\sin\frac{n\pi x}{2}$

$$=\sum_{n\ \mathrm{odd}\ \geq 1}\left(-\frac{n^2\pi^2}{4}+2\pi\right)A_n\sin\frac{n\pi x}{2}$$

于是 $\begin{cases} 2=\left(-\frac{\pi^2}{4}+2\pi\right)A_1 \\ A_n=0,\ if\ n\neq 1 \end{cases} \Rightarrow A_1=\frac{2}{-\frac{\pi^2}{4}+2\pi}=\frac{8}{8\pi-\pi^2}\Rightarrow y(x)=\frac{8}{8\pi-\pi^2}\sin\frac{\pi x}{2}.\square$

6.求解下列微分方程的特征值问题:

$$\begin{cases} y''(x)=\lambda y(x), \ 0<x<l \\ y(0)=y(l)=0 \end{cases}$$

即求那些使得上述问题有非零的实数λ及相应的非零解$y(x)$.

有非零解的实数λ叫特征值, 相应的非零解$y(x)$叫特征函数.

解:$y''(x)=\lambda y(x) \Rightarrow y''(x)y'(x)=\lambda y(x)y'(x) \Rightarrow [(y'(x))^2]'=\lambda[(y(x))^2]'$

形式上求出特解: $(y'(x))^2=\lambda(y(x))^2$

$\Rightarrow y'(x)=\sqrt{\lambda}\,y(x), y'(x)=-\sqrt{\lambda}\,y(x)$

$\Rightarrow \dfrac{y'(x)}{y(x)}=\pm\sqrt{\lambda}\,x$

$\Rightarrow \ln y(x)=\pm\sqrt{\lambda}\,x$

$\Rightarrow y(x)=e^{\pm\sqrt{\lambda}x}$,是所有可能的特解.

由于特解的线性组合依然是特解,于是全部解为$y(x)=\alpha e^{\sqrt{\lambda}x}+\beta e^{-\sqrt{\lambda}x}, \alpha,\beta\in\mathbb{C}$.(我们在复数域上考虑)

验证初值条件:$\begin{cases} y(0)=\alpha e^{\sqrt{\lambda}0}+\beta e^{-\sqrt{\lambda}0} \\ y(l)=\alpha e^{\sqrt{\lambda}l}+\beta e^{-\sqrt{\lambda}l} \end{cases} \Rightarrow \begin{cases} 0=\alpha+\beta \\ 0=\alpha e^{\sqrt{\lambda}l}+\beta e^{-\sqrt{\lambda}l} \end{cases}$

若$\lambda>0$,则$\alpha e^{\sqrt{\lambda}l}-\alpha e^{-\sqrt{\lambda}l}=0 \Rightarrow \alpha=\beta=0$或者$e^{\sqrt{\lambda}l}=e^{-\sqrt{\lambda}l} \Rightarrow y(x)=0$,无非零解

若$\lambda<0$,则$\alpha e^{i\sqrt{-\lambda}l}-\alpha e^{-i\sqrt{-\lambda}l}=\alpha\cdot 2i\sin\left(\sqrt{-\lambda}\,l\right)=0 \Rightarrow \alpha=0$或$\sin\left(\sqrt{-\lambda}\,l\right)=0$

考虑非零解,则有$\sqrt{-\lambda}\,l=k\pi, k\in\mathbb{Z}$,即$\lambda=-\dfrac{k^2\pi^2}{l^2}, k\in\mathbb{Z}$.

对应的所有非零解为:$y(x)=\alpha\left[e^{\sqrt{-\frac{k^2\pi^2}{l^2}}x}-e^{-\sqrt{-\frac{k^2\pi^2}{l^2}}x}\right]=\alpha\cdot 2i\sin\dfrac{k\pi x}{l}=\alpha'\cdot\sin\dfrac{k\pi x}{l}, k\in\mathbb{Z}, \alpha$为常数.□

例1: $f(x)$ is $2\pi-$periodic function and integrable on $[-\pi,\pi]$.

Denote the Fourier coeffecients of $f(x)$ are $a_0,a_n,b_n\,(n=1,2,\cdots)$. Show that:

(1)If $f(x)$ is monotone funciton on $(-\pi,\pi)$, then

$$|a_n|\le\frac{1}{n\pi}|f(\pi-0)-f(-\pi+0)|,|b_n|\le\frac{2}{n\pi}|f(\pi-0)-f(-\pi+0)|$$

(2)If $f(x)$ has bouned variation on $[-\pi,\pi]$, then

$$|a_n|\le\frac{1}{n\pi}\bigvee_{-\pi}^{\pi}(f),|b_n|\le\frac{2}{n\pi}\bigvee_{-\pi}^{\pi}(f)$$

Pf: (1) $|a_n|=\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx\right|\overset{f(x)\text{ is monotonic}}{=}\left|\frac{1}{\pi}\left[f(-\pi+0)\int_{-\pi}^{\xi}\cos nxdx+f(\pi-0)\int_{\xi}^{\pi}\cos nxdx\right]\right|$

$$=\left|\frac{1}{\pi}\left[f(-\pi+0)\frac{\sin n\xi}{n}-f(\pi-0)\frac{\sin n\xi}{n}\right]\right|\le\frac{1}{n\pi}|f(\pi-0)-f(-\pi+0)|$$

$$|b_n|=\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx\right|\overset{f(x)\text{ is monotonic}}{=}\left|\frac{1}{\pi}\left[f(-\pi+0)\int_{-\pi}^{\xi}\sin nxdx+f(\pi-0)\int_{\xi}^{\pi}\sin nxdx\right]\right|$$

$$=\left|\frac{1}{\pi}\left[f(-\pi+0)\frac{(-1)^n-\cos n\xi}{n}-f(\pi-0)\frac{(-1)^n-\cos n\xi}{n}\right]\right|\le\frac{2}{n\pi}|f(\pi-0)-f(-\pi+0)|$$

(2) $|a_n|=\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx\right|=\left|\frac{1}{\pi}\sum_{k=0}^{4n-1}\int_{-\pi+\frac{k}{2n}\pi}^{-\pi+\frac{k+1}{2n}\pi}f(x)\cos nxdx\right|=\left|\frac{1}{\pi}\sum_{k=0}^{4n-1}f(\xi_k)\int_{-\pi+\frac{k}{2n}\pi}^{-\pi+\frac{k+1}{2n}\pi}\cos nxdx\right|$

$$=\left|\frac{1}{n\pi}\sum_{k=0}^{4n-1}a_kf(\xi_k)\right|\le\frac{1}{n\pi}\bigvee_{-\pi}^{\pi}(f),\ \textit{where}\ \{a_k\}_{k\ge0}=\{1,-1,-1,1,1,-1,-1,1,\cdots\}$$

$$|b_n|=\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx\right|=\left|\frac{1}{\pi}\sum_{k=0}^{2n-1}\int_{-\pi+\frac{k}{n}\pi}^{-\pi+\frac{k+1}{n}\pi}f(x)\sin nxdx\right|=\left|\frac{1}{\pi}\sum_{k=0}^{2n-1}f(\zeta_k)\int_{-\pi+\frac{k}{n}\pi}^{-\pi+\frac{k+1}{n}\pi}\sin nxdx\right|$$

$$=\left|\frac{1}{\pi}\sum_{k=0}^{2n-1}f(\zeta_k)(-1)^k\cdot2\right|\le\frac{2}{n\pi}\bigvee_{-\pi}^{\pi}(f).\square$$

例2.我们首先来考虑f在$[a,b]$上黎曼可积的情况，再用光滑紧支函数逼近L^1函数完成证明.

(i)显然φ在$\mathbb{R}$上有界

不妨设$\int_0^T \varphi(x)dx=0, T=1$,否则用$\varphi(Tx)-\int_0^1 \varphi(Tx)dx$代替$\varphi(x)$

①$\forall \alpha,\beta\in\mathbb{R}$,有$\left|\int_\alpha^\beta \varphi(x)dx\right|=\left|\int_0^{\{\beta-\alpha\}} \varphi(x)dx\right|\le \sup\limits_{x\in\mathbb{R}}|\varphi(x)|$

②由于$f\in\mathscr{R}[a,b]$,则对于任意给定的$\varepsilon>0$,存在$[a,b]$上的一个划分$P=\{t_0,t_1,\cdots,t_n\}$

$a=t_0<t_1<\cdots<t_n=b$,使得$\sum\limits_{i=0}^{n-1}\left(\sup\limits_{t_i\le x\le t_{i+1}} f(x)-\inf\limits_{t_i\le x\le t_{i+1}} f(x)\right)(t_{i+1}-t_i)\le\frac{\varepsilon}{2\sup\limits_{x\in\mathbb{R}}|\varphi(x)|}$

由于$f\in\mathscr{R}[a,b]$,故$\left|\sum\limits_{i=0}^{n-1}\inf\limits_{t_i\le x\le t_{i+1}} f(x)\right|<\infty$,令$\lambda\ge\frac{2}{\varepsilon}\sup\limits_{x\in\mathbb{R}}|\varphi(x)|\left|\sum\limits_{i=0}^{n-1}\inf\limits_{t_i\le x\le t_{i+1}} f(x)\right|$

于是$\left|\int_a^b f(x)\varphi(\lambda x)dx\right|=\left|\sum\limits_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} f(x)\varphi(\lambda x)dx\right|$

$$\le\left|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left[f(x)-\inf_{t_i\le x\le t_{i+1}} f(x)\right]\varphi(\lambda x)dx\right|+\left|\sum_{i=0}^{n-1}\inf_{t_i\le x\le t_{i+1}} f(x)\int_{t_i}^{t_{i+1}}\varphi(\lambda x)dx\right|$$

$$\le\sup_{x\in\mathbb{R}}|\varphi(x)|\sum_{i=0}^{n-1}\left(\sup_{t_i\le x\le t_{i+1}} f(x)-\inf_{t_i\le x\le t_{i+1}} f(x)\right)(t_{i+1}-t_i)+\left|\sum_{i=0}^{n-1}\inf_{t_i\le x\le t_{i+1}} f(x)\frac{\int_{\lambda t_i}^{\lambda t_{i+1}}\varphi(x)dx}{\lambda}\right|$$

$$\le\sup_{x\in\mathbb{R}}|\varphi(x)|\frac{\varepsilon}{2\sup\limits_{x\in\mathbb{R}}|\varphi(x)|}+\left|\sum_{i=0}^{n-1}\inf_{t_i\le x\le t_{i+1}} f(x)\frac{\sup\limits_{x\in\mathbb{R}}|\varphi(x)|}{\frac{2}{\varepsilon}\sup\limits_{x\in\mathbb{R}}|\varphi(x)|\left|\sum\limits_{i=0}^{n-1}\inf\limits_{t_i\le x\le t_{i+1}} f(x)\right|}\right|$$

$$=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

由ε任意性: $\lim\limits_{|\lambda|\to\infty}\int_a^b f(x)\varphi(\lambda x)dx=0$.

于是(f黎曼可积的)黎曼引理得证: $\lim\limits_{|\lambda|\to\infty}\int_a^b f(x)\varphi(\lambda x)dx=\left(\frac{1}{T}\int_0^T\varphi(x)dx\right)\left(\int_a^b f(x)dx\right)$.

(ii)若$f\in L^1[a,b]$,我们宁可证明更一般的:对于任意可测集$E\subset\mathbb{R}$,g是定义在$\mathbb{R}$上的周期$T(T>0)$的有界可测函数,那么我们有

$$\lim_{|\lambda|\to\infty}\int_E f(x)\varphi(\lambda x)dx=\left(\frac{1}{T}\int_0^T\varphi(x)dx\right)\left(\int_E f(x)dx\right)$$

对于f在E外补充定义为0,于是$f\in L^1(\mathbb{R})$

我们延续(i)中的精神,不妨设$\frac{1}{T}\int_0^T\varphi(x)dx=0, T=1$,只需验证$\lim\limits_{|\lambda|\to\infty}\int_E f(x)\varphi(\lambda x)dx=0$.

由于$C_c^\infty(\mathbb{R})$函数空间在$L^1(\mathbb{R})$中稠密,于是对于任意给定的$\epsilon>0$,

存在函数$g\in C_c^\infty(\mathbb{R})$,使得$\int_{\mathbb{R}}|f(x)-g(x)|dx<\epsilon$

于是$\left|\int_{\mathbb{R}} f(x)\varphi(\lambda x)dx\right|\le\left|\int_{\mathbb{R}} g(x)\varphi(\lambda x)dx\right|+\left|\int_{\mathbb{R}}|f(x)-g(x)||\varphi(\lambda x)dx\right|\le\left|\int_{\mathbb{R}} g(x)\varphi(\lambda x)dx\right|+\epsilon\cdot\sup\limits_{x\in\mathbb{R}}|\varphi(x)|$

于是$\overline{\lim\limits_{|\lambda|\to\infty}}\left|\int_{\mathbb{R}} f(x)\varphi(\lambda x)dx\right|\le\epsilon\cdot\sup\limits_{x\in\mathbb{R}}|\varphi(x)|$

由ϵ任意性,可知$\lim\limits_{|\lambda|\to\infty}\int_E f(x)\varphi(\lambda x)dx=0$.□

(iii) 出于兴趣,我们证明对于 $f\in L^p(E)\ (p>1), \varphi(x)\in L^q[0,T]$,周期 $T>0$

其中 $E\subset\mathbb{R}$ 是有界的可测集,依然有黎曼引理成立

这里我们考虑引入Friedrichs mollifier technique.

考虑线性泛函:$T_\lambda: L^p(E)\times L^q(E)\to\mathbb{R}, (f,\varphi)\mapsto\int_E|f(x)\varphi(\lambda x)|dx$

因为 E 是有界的,所以不妨设 $E=[-mT,mT]\ (m\in\mathbb{Z})$,否则取 f 在 $[-mT,mT]-E$ 上为0即可.

继承 (i) 中的精神,我们不妨设 $\int_0^T\varphi(x)dx=0$,只需证 $\lim_{|\lambda|\to\infty}\int_E|f(x)\varphi(\lambda x)|dx=0$

由赫尔德不等式,$\forall f\in L^p(E), \varphi\in L^q(E)$,有

$$\int_E|f(x)\varphi(\lambda x)|dx\le\left(\int_{-mT}^{mT}(f(x))^p dx\right)^{\frac{1}{p}}\left(\int_{-mT}^{mT}(\varphi(\lambda x))^q dx\right)^{\frac{1}{q}}$$

$$\le\left(\int_{-mT}^{mT}(f(x))^p dx\right)^{\frac{1}{p}}\left(\frac{1}{|\lambda|}\int_{-\lambda mT}^{\lambda mT}(\varphi(x))^q dx\right)^{\frac{1}{q}}$$

$$\le\left(\int_{-mT}^{mT}(f(x))^p dx\right)^{\frac{1}{p}}\left(\frac{2[|\lambda|m]+2}{|\lambda|}\int_0^T(\varphi(x))^q dx\right)^{\frac{1}{q}}$$

$$\le\left(\int_{-mT}^{mT}(f(x))^p dx\right)^{\frac{1}{p}}\left(\int_0^T(\varphi(x))^q dx\right)^{\frac{1}{q}}(2m+2)^{\frac{1}{q}}$$

记 $\|f\|_p=\left(\int_{-mT}^{mT}(f(x))^p dx\right)^{\frac{1}{p}}, \|\varphi\|_q=\left(\int_0^T(\varphi(x))^q dx\right)^{\frac{1}{q}}$

于是 $\int_E|f(x)\varphi(\lambda x)|dx\le(2m+2)^{\frac{1}{q}}\cdot\|f\|_p\cdot\|\varphi\|_q$

于是 $|T_y(f,\varphi)|\le(2m+2)^{\frac{1}{q}}\cdot\|f\|_p\cdot\|\varphi\|_q$

对于任意给定的 $\epsilon>0$,由于 $C_c^\infty(\mathbb{R})$ 在 $L^p(\mathbb{R})$ 中稠密,所以存在 $g\in C_c^\infty(\mathbb{R})$ 和连续周期函数 ψ,使得

$\|f-g\|_p<\epsilon, \|\varphi-\psi\|_q<\epsilon$

由经典的黎曼引理可知: $\overline{\lim_{|\lambda|\to\infty}}|T_y(g,\psi)|=0$

于是 $\overline{\lim_{|\lambda|\to\infty}}|T_y(f,\varphi)|\le\overline{\lim_{|\lambda|\to\infty}}|T_y(f,\varphi-\psi)|+\overline{\lim_{|\lambda|\to\infty}}|T_y(f,\psi)|$

$\le\overline{\lim_{|\lambda|\to\infty}}|T_y(f,\varphi-\psi)|+\overline{\lim_{|\lambda|\to\infty}}|T_y(g,\psi)|+\overline{\lim_{|\lambda|\to\infty}}|T_y(f-g,\psi)|$

$\le\overline{\lim_{|\lambda|\to\infty}}(2m+2)^{\frac{1}{q}}\cdot\|f\|_p\cdot\|\varphi-\psi\|_q+\overline{\lim_{|\lambda|\to\infty}}(2m+2)^{\frac{1}{q}}\cdot\|f-g\|_p\cdot\|\varphi\|_q$

$\le\epsilon\cdot(2m+2)^{\frac{1}{q}}(\|f\|_p+\|\varphi\|_q)$

由 ϵ 任意性: $\lim_{|\lambda|\to\infty}|T_y(f,\varphi)|=0.\square$

(iv)出于兴趣,我们证明二元函数的黎曼引理.

f是定义在$[a,b]$上的可测函数,值域含于$[c,d]$,φ是可测周期$T(T>0)$的函数

且置于含于$[s,t]$,则对于$F\in C([c,d]\times[s,t])$和可测集$E\subset[a,b]$,我们有

$$\lim_{|\lambda|\to\infty}\int_E F(f(t),\varphi(\lambda t))dt=\frac{1}{T}\int_E\int_0^T F(f(x),\varphi(y))dxdy$$

由黎曼引理可知:

$$\lim_{|\lambda|\to\infty}\int_E (f(t))^i(\varphi(\lambda t))^j dt=\frac{1}{T}\int_E\int_0^T (f(x))^i(\varphi(y))^j dxdy$$

由$Stone-Weierstrass$定理可知:$C([c,d]\times[s,t])$上的函数可以由二元多项式一致地逼近

于是对于任意$\epsilon>0$,存在二元多项式$p(x,y)$使得$\sup\limits_{(x,y)\in[c,d]\times[s,t]}|p(x,y)-F(x,y)|<\epsilon$

于是$\underline{\lim}_{|\lambda|\to\infty}\int_E F(f(t),\varphi(\lambda t))dt\geq-\underline{\lim}_{|\lambda|\to\infty}\int_E|(F-p)(f(t),\varphi(\lambda t))|dt+\underline{\lim}_{|\lambda|\to\infty}\int_E p(f(t),\varphi(\lambda t))dt$

$$\geq-(b-a)\cdot\epsilon+\frac{1}{T}\int_E\int_0^T p(f(x),\varphi(y))dxdy$$

$$\geq-(b-a)\cdot\epsilon-\frac{1}{T}\int_E\int_0^T|p-F|(f(x),\varphi(y))dxdy+\frac{1}{T}\int_E\int_0^T F(f(x),\varphi(y))dxdy$$

$$\geq-2(b-a)\cdot\epsilon+\frac{1}{T}\int_E\int_0^T F(f(x),\varphi(y))dxdy$$

$$\overline{\lim}_{|\lambda|\to\infty}\int_E F(f(t),\varphi(\lambda t))dt\leq\overline{\lim}_{|\lambda|\to\infty}\int_E|(F-p)(f(t),\varphi(\lambda t))|dt+\overline{\lim}_{|\lambda|\to\infty}\int_E p(f(t),\varphi(\lambda t))dt$$

$$\leq(b-a)\cdot\epsilon+\frac{1}{T}\int_E\int_0^T p(f(x),\varphi(y))dxdy$$

$$\leq(b-a)\cdot\epsilon+\frac{1}{T}\int_E\int_0^T|p-F|(f(x),\varphi(y))dxdy+\frac{1}{T}\int_E\int_0^T F(f(x),\varphi(y))dxdy$$

$$\leq 2(b-a)\cdot\epsilon+\frac{1}{T}\int_E\int_0^T F(f(x),\varphi(y))dxdy$$

由ϵ任意性可知$\underline{\lim}_{|\lambda|\to\infty}\int_E F(f(t),\varphi(\lambda t))dt=\overline{\lim}_{|\lambda|\to\infty}\int_E F(f(t),\varphi(\lambda t))dt=\frac{1}{T}\int_E\int_0^T F(f(x),\varphi(y))dxdy$

即$\lim\limits_{|\lambda|\to\infty}\int_E F(f(t),\varphi(\lambda t))dt=\frac{1}{T}\int_E\int_0^T F(f(x),\varphi(y))dxdy.\square$