6. ~~(1). $x\in[-\pi,\pi]$ 时.~~ $f(x)=\cosh ax = \frac{e^{ax}+e^{-ax}}{2}$. ~~$f'(x)=\sinh ax=\frac{e^{ax}-e^{-ax}}{2}\cdot a$~~

~~于是 $|f'(x)|$~~

6. (1). $x\in[-\pi,\pi]$ 时. $f(x)=\cosh ax = \frac{e^{ax}+e^{-ax}}{2}$

$x\in(-\pi,\pi)$ 时. $f'(x)=a\sinh ax = a\cdot\frac{e^{ax}-e^{-ax}}{2}$

注: $f(x)$ 在 $(2k+1)\pi$ 处不可微 $(k\in\mathbb{Z})$

Check: $f(x)$ 在 $[-\pi,\pi]$ Lipschitz连续

$\forall x,y\in[-\pi,\pi]$, $(x<y)$. $\exists\xi\in(x,y)\subseteq(-\pi,\pi)$

$|f(x)-f(y)|=|f'(\xi)|\,|x-y|\le\frac{1}{2}|a|\,|e^{a\pi}-e^{-a\pi}|\,|x-y|$.

由于f是以2π为周期的函数.

故 f在 $\mathbb{R}$上 Lipschitz连续.

故 f 在$\mathbb{R}$上的 Fourier 级数收敛到 f.

$$f(x)=\frac{a_0}{2}+\sum_{n\ge1}^{\infty}a_n\cos nx+b_n\sin nx$$

于是 $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{e^{ax}+e^{-ax}}{2}dx=\frac{1}{a\pi}\int_{-\pi}^{\pi}de^{ax}=\frac{e^{a\pi}-e^{-a\pi}}{a\pi}$

$=\frac{2\sinh a\pi}{a\pi}$

~~$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{ax}+e^{-ax})\cos nx\,dx$~~

~~$=\frac{1}{\pi}\int_0^{\pi}e^{ax}\cos nx\,dx=\frac{1}{a\pi}\int_0^{\pi}\cos nx\,de^{ax}$~~

~~$=\frac{1}{a\pi}\left[(\cos n\pi\, e^{a\pi}-1)-\int_0^{\pi}e^{ax}d\cos nx\right]$~~

~~$=\frac{1}{a\pi}\left[((-1)^n e^{a\pi}-1)+n\int_0^{\pi}e^{ax}\sin nx\,dx\right]$~~

~~$=\frac{(-1)^n e^{a\pi}-1}{a\pi}+\frac{n}{a^2\pi}\int_0^{\pi}\sin nx\,de^{ax}$~~

~~$=\frac{(-1)^n e^{a\pi}-1}{a\pi}+\frac{-n^2}{a^2\pi}\int_0^{\pi}e^{ax}\cos nx\,dx$~~

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{e^{ax}+e^{-ax}}{2}\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}e^{ax}\cos nx\,dx = \frac{1}{a\pi}\int_{-\pi}^{\pi}\cos nx\,de^{ax}$$

$$= \frac{1}{a\pi}\left[\left(e^{a\pi}\cos n\pi - e^{-a\pi}\cos(-n\pi)\right) + n\int_{-\pi}^{\pi}e^{ax}\sin nx\,dx\right]$$

$$= \frac{\sinh ax}{a\pi}(-1)^n + \frac{n}{a^2\pi}\int_{-\pi}^{\pi}\sin nx\,de^{ax}$$

$$= \frac{\sinh ax}{a\pi}(-1)^n - \frac{n^2}{a^2\pi}\int_{-\pi}^{\pi}e^{ax}\cos nx\,dx$$

$$= \frac{2\sinh ax}{\pi}\cdot\frac{(-1)^n\cdot a}{n^2+a^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cosh ax\sin nx\,dx = 0.$$

$$\Rightarrow f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{a\cos nx}{n^2+a^2}.\ \forall x\in\mathbb{R}.\ \cdots(*)$$

(2). 在(*)式中令. $x=\pi$. 则有

$$\cosh a\pi = \frac{\sinh a\pi}{a\pi} + \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{a}{n^2+a^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = -\frac{1}{2a^2} + \frac{\pi}{2a}\coth a\pi.\ (\forall a>0)$$

(3). 在(*)式中令 $x=0$. 则有

$$\cosh 0 = \frac{\sinh a\pi}{a\pi} + \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{a\cdot(-1)^n}{n^2+a^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+a^2} = \frac{1}{2a^2} - \frac{\pi}{2a\sinh a\pi}.\ (\forall a>0)$$

□

~~早.(1).~~

7. (1). $f(x)$ 是 2π 周期函数，在 $(-\pi,\pi]$ 上等于 $\sinh ax$，其中 a 是常数

显然 $f(x)$ 在 $(2k+1)\pi$ 处间断 ($k\in\mathbb{Z}$).

$x\in(-\pi,\pi)$ 时，$f'(x)=a\cosh ax$.

$\forall x,y\in(-\pi,\pi)$，$x<y$，$\exists\,\xi\in(x,y)$，使得

$|f(x)-f(y)|=|f'(\xi)|\,|x-y|\le|a\cosh a\pi|\,|x-y|$

故 f 在 $(-\pi,\pi)$ 上 Lipschitz 连续

故 f 在 $\mathbb{R}\setminus\{(2m-1)\pi\mid m\in\mathbb{Z}\}$ 上 Lipschitz 连续

故 f 在 $\mathbb{R}\setminus\{(2m-1)\pi\mid m\in\mathbb{Z}\}$ 上 Fourier 级数收敛到 f

故 $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\sinh ax\,dx=-\frac{1}{\pi}\int_{-\pi}^{\pi}\sinh ax\,dx=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\sinh ax\cos nx\,dx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\sinh ax\sin nx\,dx=\frac{2\sinh a\pi}{\pi}\cdot\frac{n\cdot(-1)^{n-1}}{n^2+a^2}$$

于是 $f(x)=\frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n\sin nx}{n^2+a^2}$，$\forall x\in\mathbb{R}\setminus\{(2m-1)\pi\mid m\in\mathbb{Z}\}$ $\cdots(*)$

(3). 在 $(*)$ 式中取 $x=\frac{\pi}{2}$，就有

$$\sinh\frac{a\pi}{2}=f\left(\frac{\pi}{2}\right)=\frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n\sin\frac{n\pi}{2}}{n^2+a^2}$$

其中 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n\sin\frac{n\pi}{2}}{n^2+a^2}=\frac{1}{1^2+a^2}+0-\frac{3}{3^2+a^2}+0+\frac{5}{5^2+a^2}+0-\frac{7}{7^2+a^2}+0+\cdots$

于是找规律发现 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n\sin\frac{n\pi}{2}}{n^2+a^2}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(2n-1)^2+a^2}$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(2n-1)^2+a^2}=\frac{\pi\sinh\frac{a\pi}{2}}{2\sinh a\pi}$$

□

9.(1) 证: $\displaystyle\int_{-\pi}^{\pi} f(x+t)\,F_n(t)\,dt = \int_{-\pi}^{\pi} f(x+t)\cdot\frac{1}{2n\pi}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt$

注意到: $D_N(t) = \dfrac{\sin\left(\left(N+\frac{1}{2}\right)t\right)}{\sin\frac{t}{2}}$

$$\frac{D_0(t)+\cdots+D_{N-1}(t)}{N} = \frac{\sin\frac{t}{2}+\cdots+\sin\left(\left(N-\frac{1}{2}\right)t\right)}{N\sin\frac{t}{2}}$$

$$= \frac{\sin\frac{t}{2}\sin\frac{t}{2}+\cdots+\sin\left(\left(N-\frac{1}{2}\right)t\right)\sin\frac{t}{2}}{N\sin^2\frac{t}{2}}$$

$$= \frac{-\cos Nt+1}{2N\sin^2\frac{t}{2}}$$

$$= \frac{\sin^2\frac{Nt}{2}}{N\sin^2\frac{t}{2}}$$

于是 $\displaystyle\int_{-\pi}^{\pi} f(x+t)\cdot\frac{1}{2n\pi}\cdot\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\,\frac{D_0(t)+\cdots+D_{n-1}(t)}{n}\,dt$$

$$= \frac{1}{2\pi}\cdot\frac{\int_{-\pi}^{\pi} f(x+t)D_0(t)\,dt+\cdots+\int_{-\pi}^{\pi} f(x+t)D_{n-1}(t)\,dt}{n}$$

$$= \frac{S_0(x)+\cdots+S_{n-1}(x)}{n}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x+t)\,F_{n+1}(t)\,dt = \frac{S_0(x)+\cdots+S_n(x)}{n+1} = \sigma_n(x)$$

(2) (i) $F_n(t) = \dfrac{1}{2n\pi}\left(\dfrac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 \geq 0$

(ii) $\displaystyle\int_{-\pi}^{\pi} F_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{D_0(t)+\cdots+D_{n-1}(t)}{n}\,dt$

$$\cancel{\int_{-\pi}^{\pi} D_k(t)\,dt = \int_{-\pi}^{\pi}\frac{\sin\left(k\cdot\frac{t}{2}\right)}{\sin\frac{t}{2}}\,dt}$$

$$\cancel{= \int_{-\pi}^{\pi}\frac{\sin\left(\frac{k-1}{2}t\right)\cos\frac{t}{2}+\sin\frac{t}{2}\cos\left(\frac{k-1}{2}t\right)}{\sin\frac{t}{2}}\,dt}$$

$$\cancel{= \int_{-\pi}^{\pi}\left[\sin\frac{(k-1)t}{2}\cot\frac{t}{2}+\cos\frac{(k-1)t}{2}\right]dt}$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\sin((k-1)t)\cot t+\cos((k-1)t)\right]dt$$

$$\cancel{\int_{-\pi}^{\pi} D_k(x)\,dx = \int_{-\pi}^{\pi} \frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\,dx = 4\int_0^{\frac{\pi}{2}} \frac{\sin(nx)}{\sin x}\,dx = 4\int_0^{\frac{\pi}{2}} \frac{\sin((n-1)x)\cos x + \sin x\cos((n-1)x)}{\sin x}\,dx}$$

$$\cancel{= 4\int_0^{\frac{\pi}{2}} \frac{\sin((n-1)x)\cos x}{\sin x}\,dx + 4\int_0^{\frac{\pi}{2}} \cos((n-1)x)\,dx}$$

$$\int_{-\pi}^{\pi} D_k(x)\,dx = \int_{-\pi}^{\pi} \frac{\sin((n+\frac{1}{2})x)}{\sin\frac{x}{2}}\,dx = 4\int_0^{\frac{\pi}{2}} \frac{\sin((2n+1)x)}{\sin x}\,dx$$

注意到 $\cos 2x + \cos 4x + \dots + \cos(2nx) = \dfrac{\sin x\cos 2x + \dots + \sin x\cos(2nx)}{\sin x}$

$$= \frac{\sin 3x - \sin x + \sin 5x - \sin 3x + \dots + \sin((2n+1)x) - \sin((2n-1)x)}{2\sin x}$$

$$= \frac{\sin((2n+1)x) - \sin x}{2\sin x} = \frac{\sin((2n+1)x)}{2\sin x} - \frac{1}{2}.$$

于是 $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin((2n+1)x)}{\sin x}\,dx = \int_0^{\frac{\pi}{2}} (2[\cos 2x + \cos 4x + \dots + \cos(2nx)] + 1)\,dx$

$$= \frac{\pi}{2} + \sum_{k=1}^{n} 2\int_0^{\frac{\pi}{2}} \cos(2kx)\,dx$$

$$= \frac{\pi}{2} + 2\sum_{k=1}^{n} \int_0^{\frac{\pi}{2}} \frac{d\sin(2kx)}{2k}$$

$$= \frac{\pi}{2}$$

于是 $\displaystyle\int_{-\pi}^{\pi} D_k(x)\,dx = 4\int_0^{\frac{\pi}{2}} \frac{\sin((2n+1)x)}{\sin x}\,dx = 2\pi$，$\forall k$.

$$\Rightarrow \int_{-\pi}^{\pi} F_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{D_0(t) + \dots + D_{n-1}(t)}{n}\,dt = 1$$

(iii) ~~$\forall\, 0 < \delta \le \frac{\pi}{2}$，有 $\sin\delta \ge \frac{2}{\pi}\delta$.~~ 对于任意给定的 $\delta \in (0, \frac{\pi}{2})$

~~$\forall\, \frac{\pi}{2} < \delta < \pi$，有~~

~~$\forall\, 0 < \delta < \frac{\pi}{2}$. 有 $\sin\delta \ge \frac{2}{\pi}\delta$~~

$$\int_\delta^\pi F_n(t)\,dt = \frac{1}{2n\pi}\int_\delta^\pi \frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\,dt \le \frac{1}{2n\pi}\int_\delta^\pi \frac{1}{\sin^2\delta}\,dt = \frac{\pi - \delta}{2\pi n\cdot\sin^2\delta}$$

故 $\displaystyle 0 \le \varliminf_{n\to\infty} \int_\delta^\pi F_n(t)\,dt \le \varlimsup_{n\to\infty} \frac{\pi - \delta}{2\pi n\cdot\sin^2\delta} = 0.$

$$\lim_{n\to\infty}\int_\delta^\pi F_n(t)\,dt = 0$$

故 $\forall\, 0 < \delta < \frac{\pi}{2}$，有 $\displaystyle\lim_{n\to\infty}\int_\delta^\pi F_n(t)\,dt = 0$

$\forall\, \delta \in [\frac{\pi}{2}, \pi)$. ~~有~~ 由于 $F_n(t) \ge 0$. 故 $\displaystyle 0 \le \varliminf_{n\to\infty}\int_\delta^\pi F_n(t)\,dt \le \varlimsup_{n\to\infty}\int_{\frac{\pi}{4}}^\pi F_n(t)\,dt = 0$

故 $\forall\, \delta \in (0, \pi)$. 有 $\displaystyle\lim_{n\to\infty}\int_\delta^\pi$

(3). 对于每个使 $f(x)$ 连续的点 $x_0\in(-\pi,\pi)$. 由连续性定义:

$\forall \varepsilon>0,\ \exists \delta>0$ s.t. $|f(x)-f(x_0)|<\varepsilon$. $\forall x\in(x_0-\delta, x_0+\delta)\subseteq(-\pi,\pi)$

于是 $|\sigma_n(x)-f(x_0)| = \left|\int_{-\pi}^{\pi} f(x+t)F_{n+1}(t)\,dt - f(x_0)\right|$

$$= \left|\int_{-\pi}^{\pi}[f(x+t)-f(x_0)]F_{n+1}(t)\,dt\right| \quad \left(\text{由于} \int_{-\pi}^{\pi}F_{n+1}(t)\,dt=1\right)$$

$$\le \int_{-\delta/2}^{\delta/2}|f(x+t)-f(x_0)|\,F_{n+1}(t)\,dt + \int_{\pi\ge|t|\ge\frac{\delta}{2}}|f(x+t)-f(x_0)|$$

(是对的)

(3). $\forall \varepsilon>0,\ \exists\delta>0$ s.t. $|f(x)-f(x_0)|<\varepsilon$. $\forall x\in(x_0-\delta, x_0+\delta)$

$$\sigma_n(x)-f(x) = \int_{-\pi}^{\pi} f(x+t)F_{n+1}(t)\,dt - f(x)$$

$$= \int_{-\pi}^{\pi}[f(x+t)-f(x)]\,F_{n+1}(t)\,dt$$

$$= \int_{[-\frac{\delta}{2},\frac{\delta}{2}]}[f(x+t)-f(x)]\,F_{n+1}(t)\,dt$$

$$+ \int_{[-\pi,-\frac{\delta}{2})\cup(\frac{\delta}{2},\frac{\pi}{2}]}[f(x+t)-f(x)]\,F_{n+1}(t)\,dt$$

(3). ① lemma: $f * F_{n+1}$ 连续(在$[-\pi,\pi]$上). $\forall n$.

于是 $\lim\limits_{x\to x_0}\sigma_n(x) = \sigma(x_0)$

Check: $\sigma(x_0) = \int_{-\pi}^{\pi} f(x_0+t)F_{n+1}(t)\,dt = f(x_0)$. (as $n\to\infty$)

$\forall \varepsilon>0$, $\exists \delta>0$ (足够小), s.t. $\forall |x-x_0|<\delta$, 有

$$|f(x)-f(x_0)|<\varepsilon$$

$$\left|\int_{-\pi}^{\pi} f(x_0+t)F_{n+1}(t)\,dt - f(x_0)\right|$$

$$= \left|\int_{-\pi}^{\pi}[f(x_0+t)-f(x_0)]F_{n+1}(t)\,dt\right| \quad \left(\text{由于} \int_{-\pi}^{\pi}F_{n+1}(t)\,dt = 1\right)$$

$$= \left|\int_{(-\delta,\delta)}[f(x_0+t)-f(x_0)]F_{n+1}(t)\,dt + \int_{[-\pi,-\delta]\cup[\delta,\pi]}[f(x_0+t)-f(x_0)]F_{n+1}(t)\,dt\right|$$

$$\le \int_{(-\delta,\delta)}|f(x_0+t)-f(x_0)|F_{n+1}(t)\,dt \cancel{+ \int_{[-\pi,-\delta]\cup[\delta,\pi]} f(x_0+t)F_{n+1}(t)\,dt}$$

$$+ 2\sup_{-\pi\le x\le\pi}|f(x)|\int_{\delta\le|t|\le\pi}F_{n+1}(t)\,dt$$

$$\le \varepsilon\int_{-\pi}^{\pi}F_{n+1}(t)\,dt + 2\sup_{-\pi\le x\le\pi}|f(x)|\int_{\delta\le|t|\le\pi}F_{n+1}(t)\,dt$$

$$= \varepsilon + 2\sup_{-\pi\le x\le\pi}|f(x)|\int_{\delta\le|t|\le\pi}F_{n+1}(t)\,dt.$$

于是 $0 \le \varliminf\limits_{n\to\infty}\left|\int_{-\pi}^{\pi}f(x_0+t)F_{n+1}(t)\,dt - f(x_0)\right| \le \varlimsup\limits_{n\to\infty}\left(\varepsilon + 2\sup\limits_{|x|\le\pi}|f(x)|\int_{\delta\le|t|\le\pi}F_{n+1}(t)\,dt\right) = \varepsilon$

由 ε 任意性可知: $\lim\limits_{n\to\infty}\int_{-\pi}^{\pi}f(x_0+t)F_{n+1}(t)\,dt = f(x_0)$.

② ~~若 $f(x)$ 是连续的 2π 周期函数. 则~~ ~~$f(x)$ 在 $[-\pi,\pi]$ 上一致连续, 则 $\forall\varepsilon>0$, $\exists\delta>0$ 使 s.t. $\forall x, y\in[-\pi,\pi]$, $|x-y|<\delta$ 有~~

~~$|f(x)-f(y)|<\varepsilon$.~~

~~于是 $\forall x\in[-\pi,\pi]$.~~ 因为

③ 若 f 是连续的，以 2π 为周期的函数

则 f 在 $[-2\pi, 2\pi]$ 上一致连续，

$\forall \varepsilon>0$. $\exists \delta>0$. s.t. $\forall x, y \in [-2\pi, 2\pi]$，且 $|x-y|<\delta$. 有

$$|f(x)-f(y)|<\varepsilon$$

由于 $\lim\limits_{n\to\infty} \int_{\delta\le t\le\pi} F_n(t)\,dt = 0$. 故 $\exists N>0$. s.t. $\forall n>N$. 有

$$\left|\int_{\delta\le t\le\pi} F_{n+1}(t)\,dt\right| < \varepsilon.$$

对称地，$\left|\int_{-\pi\le t\le-\delta} F_{n+1}(t)\,dt\right| < \varepsilon$

于是 $|\sigma_n(x)-f(x)| = \left|\int_{-\pi}^{\pi} (f(x+t)-f(x))\,F_{n+1}(t)\,dt\right|$

$$\le \int_{-\delta}^{\delta} |f(x+t)-f(x)|\cdot F_{n+1}(t)\,dt + 2\sup_{x\in[-\pi,\pi]}|f(x)|\int_{\delta\le|t|\le\pi} F_{n+1}(t)\,dt$$

$$\le \varepsilon + 2\sup_{-\pi\le x\le\pi}|f(x)|\cdot(2\varepsilon) = \left(4\sup_{-\pi\le x\le\pi}|f(x)|+1\right)\varepsilon$$

由 ε 任意性：$\lim\limits_{n\to\infty}|\sigma_n(x)-f(x)| = 0$ #

(4) 由于 Fejér 核是 Good Kernel，所以 $f*F_{m+1}$ 在 x_0 处的 Fourier 级数收敛到 $(f*F_{m+1})(x_0)$. i.e. $\sum\limits_{n=-\infty}^{\infty} \widehat{f*F_{m+1}}(n)\, e^{inx_0} \longrightarrow (f*F_{m+1})(x_0)$

又因为 $\begin{cases} \widehat{f*F_{m+1}}(n) = \hat{f}(n)\,\hat{F}_{m+1}(n) = \hat{f}(n) \\ f*F_{m+1}(x_0) \longrightarrow f(x_0) \quad (\text{as } m\to\infty) \end{cases}$

故 $\sum\limits_{n=-\infty}^{\infty} \hat{f}(n)\, e^{inx_0} \longrightarrow f(x_0)$

故 f 在 x_0 处的 Fourier 级数收敛到 $f(x_0)$. □

10.显然φ在$\mathbb{R}$上有界

不妨设$\int_0^T \varphi(x)dx=0, T=1$,否则用$\varphi(Tx)-\int_0^1 \varphi(Tx)dx$代替$\varphi(x)$

①$\forall \alpha,\beta\in\mathbb{R}$,有$\left|\int_\alpha^\beta \varphi(x)dx\right|=\left|\int_0^{\{\beta-\alpha\}} \varphi(x)dx\right|\le \sup\limits_{x\in\mathbb{R}}|\varphi(x)|$

②由于$f\in\mathscr{R}[a,b]$,则对于任意给定的$\varepsilon>0$,存在$[a,b]$上的一个划分$P=\{t_0,t_1,\cdots,t_n\}$

$a=t_0<t_1<\cdots<t_n=b$,使得$\sum\limits_{i=0}^{n-1}\left(\sup\limits_{t_i\le x\le t_{i+1}} f(x)-\inf\limits_{t_i\le x\le t_{i+1}} f(x)\right)(t_{i+1}-t_i)\le \dfrac{\varepsilon}{2\sup\limits_{x\in\mathbb{R}}|\varphi(x)|}$

由于$f\in\mathscr{R}[a,b]$,故$\left|\sum\limits_{i=0}^{n-1}\inf\limits_{t_i\le x\le t_{i+1}} f(x)\right|<\infty$,令$\lambda\ge\dfrac{2}{\varepsilon}\sup\limits_{x\in\mathbb{R}}|\varphi(x)|\left|\sum\limits_{i=0}^{n-1}\inf\limits_{t_i\le x\le t_{i+1}} f(x)\right|$

于是$\left|\int_a^b f(x)\varphi(\lambda x)dx\right|=\left|\sum\limits_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} f(x)\varphi(\lambda x)dx\right|$

$$\le\left|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left[f(x)-\inf_{t_i\le x\le t_{i+1}} f(x)\right]\varphi(\lambda x)dx\right|+\left|\sum_{i=0}^{n-1}\inf_{t_i\le x\le t_{i+1}} f(x)\int_{t_i}^{t_{i+1}}\varphi(\lambda x)dx\right|$$

$$\le\sup_{x\in\mathbb{R}}|\varphi(x)|\sum_{i=0}^{n-1}\left(\sup_{t_i\le x\le t_{i+1}} f(x)-\inf_{t_i\le x\le t_{i+1}} f(x)\right)(t_{i+1}-t_i)+\left|\sum_{i=0}^{n-1}\inf_{t_i\le x\le t_{i+1}} f(x)\frac{\int_{\lambda t_i}^{\lambda t_{i+1}}\varphi(x)dx}{\lambda}\right|$$

$$\le\sup_{x\in\mathbb{R}}|\varphi(x)|\frac{\varepsilon}{2\sup\limits_{x\in\mathbb{R}}|\varphi(x)|}+\left|\sum_{i=0}^{n-1}\inf_{t_i\le x\le t_{i+1}} f(x)\frac{\sup\limits_{x\in\mathbb{R}}|\varphi(x)|}{\frac{2}{\varepsilon}\sup\limits_{x\in\mathbb{R}}|\varphi(x)|\left|\sum\limits_{i=0}^{n-1}\inf\limits_{t_i\le x\le t_{i+1}} f(x)\right|}\right|$$

$$=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

由ε任意性:$\lim\limits_{|\lambda|\to\infty}\int_a^b f(x)\varphi(\lambda x)dx=0$.

于是黎曼引理得证:$\lim\limits_{|\lambda|\to\infty}\int_a^b f(x)\varphi(\lambda x)dx=\left(\dfrac{1}{T}\int_0^T\varphi(x)dx\right)\left(\int_a^b f(x)dx\right)$

11.(2) $\sum\limits_{n=0}^{\infty}(-1)^n\int_0^1\dfrac{\sin\pi x}{n+x}\mathrm{d}x=\sum\limits_{n=0}^{\infty}\int_n^{n+1}(-1)^n\dfrac{\sin\pi(x-n)}{x}\mathrm{d}x=\sum\limits_{n=0}^{\infty}\int_n^{n+1}\dfrac{\sin\pi x}{x}\mathrm{d}x=\int_0^{\infty}\dfrac{\sin\pi x}{x}dx=\dfrac{\pi}{2}$

3.①$f(x)\sin x$

$$a_0'=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin x dx=b_1$$

$$a_n'=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin x\cos nx dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\frac{\sin((n+1)x)-\sin((n-1)x)}{2}dx=\frac{b_{n+1}-b_{n-1}}{2}$$

$$b_n'=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin x\sin nx dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\frac{\cos((n-1)x)-\cos((n+1)x)}{2}dx=\frac{a_{n-1}-a_{n+1}}{2}$$

于是$f(x)\sin x\sim\frac{b_1}{2}+\sum_{n=1}^{\infty}\frac{b_{n+1}-b_{n-1}}{2}\cos nx+\frac{a_{n-1}-a_{n+1}}{2}\sin nx$

②$f(x)\cos x$

$$a_0'=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos x dx=a_1$$

$$a_n'=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos x\cos nx dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\frac{\cos((n+1)x)+\cos((n-1)x)}{2}dx=\frac{a_{n+1}+a_{n-1}}{2}$$

$$b_n'=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos x\sin nx dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\frac{\sin((n+1)x)+\sin((n-1)x)}{2}dx=\frac{b_{n+1}+b_{n-1}}{2}$$

于是$f(x)\cos x\sim\frac{a_1}{2}+\sum_{n=1}^{\infty}\frac{a_{n+1}+a_{n-1}}{2}\cos nx+\frac{b_{n+1}+b_{n-1}}{2}\sin nx$

其中$b_0=0$.

(2)$f(x)=x\cos x$

其中$x\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x dx=0,a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos nx dx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nx dx=\frac{2}{\pi}\int_0^{\pi}x\sin nx dx=-\frac{2}{n\pi}\int_0^{\pi}x d\cos nx=-\frac{2}{n\pi}(\pi\cos n\pi-0)+\frac{2}{n\pi}\int_0^{\pi}\cos nx dx=\frac{2(-1)^{n+1}}{n}$$

于是,$x\sim\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$

于是$x\cos x\sim-\frac{\sin x}{2}+\sum_{n=2}^{\infty}\frac{\frac{2(-1)^{n+2}}{n+1}+\frac{2(-1)^{n}}{n-1}}{2}\sin nx=-\frac{\sin x}{2}+\sum_{n=2}^{\infty}(-1)^n\frac{2n}{n^2-1}\sin nx$

$(4)\, f(x)=|\sin x|$, 考虑 $g(x):=\begin{cases} -1 & \text{当} -\pi < x \le 0 \\ 1 & \text{当}\, 0 < x \le \pi \end{cases}$

$$g(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\,dx = 0,\, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{0} g(x)\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} g(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} \sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} \sin nx dx = -\frac{2}{n\pi}\int_{0}^{\pi} d\cos nx = -\frac{2}{n\pi}(\cos n\pi - 1)$$

$$= \frac{2}{n\pi}[1-(-1)^n]$$

于是, $g(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}[1-(-1)^n]\sin nx$

于是 $|\sin x| = g(x)\sin x \sim \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{\frac{2}{(n+1)\pi}[1-(-1)^{n+1}] - \frac{2}{(n-1)\pi}[1-(-1)^{n-1}]}{2}\cos nx$

$$= \frac{2}{\pi} + \sum_{n=1}^{\infty}[1+(-1)^n]\frac{-2}{(n^2-1)\pi}\cos nx$$

(6) $f(x)=|x|\cos x$

$$|x|\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|dx=\pi$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos nxdx=\frac{2}{\pi}\int_0^{\pi}x\cos nxdx=\frac{2}{n\pi}\int_0^{\pi}xd\sin nx$$

$$=-\frac{2}{n\pi}\int_0^{\pi}\sin nxdx=\frac{2}{n^2\pi}\int_0^{\pi}d\cos nx=\frac{2}{n^2\pi}[(-1)^n-1]$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin nxdx=0$$

于是, $|x|\sim\dfrac{\pi}{2}+\displaystyle\sum_{n=1}^{\infty}\frac{2}{n^2\pi}[(-1)^n-1]\cos nx$

$$|x|\cos x\sim-\frac{2}{\pi}+\sum_{n=2}^{\infty}\frac{\frac{2}{(n+1)^2\pi}[(-1)^n-1]+\frac{2}{(n-1)^2\pi}[(-1)^{n-1}-1]}{2}\cos nx$$

$$=-\frac{2}{\pi}+\sum_{n=2}^{\infty}\left[\frac{1}{(n+1)^2\pi}+\frac{1}{(n-1)^2\pi}\right][(-1)^{n+1}-1]\cos nx$$

$$=-\frac{2}{\pi}+\sum_{n=2}^{\infty}\frac{2n^2+2}{(n^2-1)^2\pi}[(-1)^{n+1}-1]\cos nx$$

(8) $f(x)=x\sin 2x=2x\cos x\sin x$

由(2)可知: $x\cos x\sim-\dfrac{\sin x}{2}+\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{2n}{n^2-1}\sin nx$

于是 $x\cos x\sin x\sim-\dfrac{1}{4}+\dfrac{2}{3}\cos x+\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}\frac{2n+2}{(n+1)^2-1}-(-1)^{n-1}\frac{2n-2}{(n-1)^2-1}}{2}\cos nx$

$$=-\frac{1}{4}+\frac{2}{3}\cos x+\sum_{n=2}^{\infty}(-1)^{n+1}\left[\frac{n+1}{(n+1)^2-1}-\frac{n-1}{(n-1)^2-1}\right]\cos nx$$

$$=-\frac{1}{4}+\frac{2}{3}\cos x+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{(n^2-1)(n-1)-(n+1)-[(n^2-1)(n+1)-(n-1)]}{n^2(n^2-4)}\cos nx$$

$$=-\frac{1}{4}+\frac{2}{3}\cos x+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{-2(n^2-1)-2}{n^2(n^2-4)}\cos nx$$

$$=-\frac{1}{4}+\frac{2}{3}\cos x+\sum_{n=2}^{\infty}(-1)^n\frac{2}{n^2-4}\cos nx$$

于是 $f(x)=x\sin 2x\sim-\dfrac{1}{2}+\dfrac{4}{3}\cos x+\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{4}{n^2-4}\cos nx$

1.证:若f,g的傅里叶级数相等,则f,g几乎处处相等

记$h=f-g$,则$\frac{1}{\pi}\int_{-\pi}^{\pi}h(x)dx=0$, $\frac{1}{\pi}\int_{-\pi}^{\pi}h(x)\cos nxdx=0$, $\frac{1}{\pi}\int_{-\pi}^{\pi}h(x)\sin nxdx=0$.

由于h绝对可积,所以可以被连续函数一致逼近,所以可以被三角多项式$T_n(x)$一致逼近($Stone-Weiestrass$定理).

但是$0=\lim_{n\to\infty}\int_{-\pi}^{\pi}h(x)T_n(x)dx=\int_{-\pi}^{\pi}h^2(x)dx$,于是$h(x)$几乎处处为0.

于是f和g几乎处处相等.

5.由狄利克雷判别法可知:$\sum_{n=1}^{\infty}a_n\sin nx$收敛

$$\left|\sum_{n=1}^{m}\sin nx\right|=\left|\frac{\sum_{n=1}^{m}\sin\frac{x}{2}\sin nx}{\sin\frac{x}{2}}\right|=\left|\frac{\sum_{n=1}^{m}\cos\frac{(n-1)x}{2}-\cos\frac{(n+1)x}{2}}{2\sin\frac{x}{2}}\right|=\left|\frac{1-\cos\frac{(m+1)x}{2}}{2\sin\frac{x}{2}}\right|$$

取$x=\frac{1}{m+1}$,则$\lim_{m\to\infty}\left|\frac{1-\cos\frac{(m+1)x}{2}}{2\sin\frac{x}{2}}\right|=\lim_{m\to\infty}\left|\frac{1-\cos\frac{1}{2}}{2\sin\frac{1}{2(m+1)}}\right|\to\infty$,并非一致有界

但是我们考虑任何给定的,与$\{2k\pi:k\in\mathbb{Z}\}$无交的有界闭区间$[a,b]$

在$[a,b]$上,$\sum_{n=1}^{m}\sin nx$一致有界,故$\sum_{n=1}^{\infty}a_n\sin nx$在$[a,b]$一致收敛.

于是和函数在$[a,b]$上连续. 由于连续是一个'逐点'的性质,所以$\sum_{n=1}^{\infty}a_n\sin nx$在$\mathbb{R}\sim\{2k\pi:k\in\mathbb{Z}\}$上连续.

反例:$\sum_{n=1}^{\infty}\frac{2\sin nx}{n}$在$x=2k\pi$处不连续($k\in\mathbb{Z}$),因为$f(x)=\begin{cases}\pi-x & x\in(0,\pi)\\ x-\pi & x\in[-\pi,0)\\ 0 & x=0\\ \text{周期延拓} & x\notin[-\pi,\pi)\end{cases}\sim\sum_{n=1}^{\infty}\frac{2\sin nx}{n}$