$1.(1)\cos^2 x\sin 2x=\dfrac{\cos 2x+1}{2}\sin 2x=\dfrac{1}{2}\sin 2x+\dfrac{1}{4}\sin 4x$

$=\dfrac{1}{2}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)!}(2x)^{2n+1}+\dfrac{1}{4}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)!}(4x)^{2n+1}=\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)!}(2^{2n}+2^{4n})x^{2n+1}$,收敛域为$\mathbb{R}$

$(4)\dfrac{x}{\sqrt{1-2x}}=x(1-2x)^{-\frac{1}{2}}=x\left(1+\sum_{n=1}^{\infty}C^n_{-\frac{1}{2}}(-2x)^n\right)=x\left[1+\sum_{n=1}^{\infty}\dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}(-2x)^n\right]$

$=x\left[1+\sum_{n=1}^{\infty}\dfrac{(2n-1)!!}{n!\,2^n}x^n\right]=x+\sum_{n=1}^{\infty}\dfrac{(2n-1)!!}{(2n)!!}x^{n+1}$,收敛域为$\left[-\dfrac{1}{2},\dfrac{1}{2}\right)$

$(5)\ln\sqrt{\dfrac{1+x^2}{1-x^2}}=\dfrac{1}{2}(\ln(1+x^2)-\ln(1-x^2))=\dfrac{1}{2}\left[\sum_{n=0}^{\infty}\dfrac{(-1)^n}{n+1}(x^2)^{n+1}-\sum_{n=0}^{\infty}\dfrac{(-1)^n}{n+1}(-x^2)^{n+1}\right]$

$=\dfrac{1}{2}\left[\sum_{n=0}^{\infty}\dfrac{(-1)^n}{n+1}x^{2n+2}+\sum_{n=0}^{\infty}\dfrac{1}{n+1}x^{2n+2}\right]=\dfrac{1}{2}\sum_{n=0}^{\infty}\dfrac{(-1)^n+1}{n+1}x^{2n+2}=\sum_{n=0}^{\infty}\dfrac{1}{2n+1}x^{4n+2}$,收敛域为$(-1,1)$

$(6)\dfrac{x}{1+x-2x^2}=\dfrac{x}{(1+2x)(1-x)}=\dfrac{1}{3}\left(\dfrac{1}{1-x}-\dfrac{1}{1+2x}\right)=\dfrac{1}{3}\left(\sum_{n=0}^{\infty}x^n-\sum_{n=0}^{\infty}(-2x)^n\right)=\sum_{n=0}^{\infty}\dfrac{1-(-2)^n}{3}x^n$

收敛域为$\left(-\dfrac{1}{2},\dfrac{1}{2}\right)$

$(9)\dfrac{x}{1+x+x^2}=\dfrac{x(1-x)}{1-x^3}=(x-x^2)\sum_{n=0}^{\infty}(x^3)^n=\sum_{n=0}^{\infty}x^{3n+1}-\sum_{n=0}^{\infty}x^{3n+2}$

$=\sum_{n=0}^{\infty}\left(\left[\dfrac{n+2}{3}\right]-\left[\dfrac{n+1}{3}\right]\right)x^n-\sum_{n=0}^{\infty}\left(\left[\dfrac{n+1}{3}\right]-\left[\dfrac{n}{3}\right]\right)x^n$

$=\sum_{n=0}^{\infty}\left(\left[\dfrac{n+2}{3}\right]-2\left[\dfrac{n+1}{3}\right]+\left[\dfrac{n}{3}\right]\right)x^n$,收敛域为$[-1,1)$

$3.(3)f(x)=\dfrac{2}{\sqrt{3}}\arctan\dfrac{2x-1}{\sqrt{3}}$

$f'(x)=\dfrac{1}{1-x+x^2}=:\sum_{n=0}^{\infty}a_nx^n\Rightarrow\sum_{n=0}^{\infty}a_nx^n-\sum_{n=0}^{\infty}a_nx^{n+1}+\sum_{n=0}^{\infty}a_nx^{n+2}=1$

$$\begin{cases}a_0=1\\a_1-a_0=0\\a_n-a_{n-1}+a_{n-2}=0,\forall n\geq 2\end{cases},\text{于是}a_n=\begin{cases}1,n\equiv 0,1\pmod 6\\0,n\equiv 2,5\pmod 6\\-1,n\equiv 3,4\pmod 6\end{cases}$$

于是$f(x)=f(0)+\int_0^x f'(t)dt=f(0)+\int_0^x\sum_{n=0}^{\infty}a_nt^ndt=f(0)+\sum_{n=0}^{\infty}a_n\int_0^x t^ndt=-\dfrac{\pi}{3\sqrt{3}}+\sum_{n=0}^{\infty}\dfrac{a_n}{n+1}x^{n+1}$

收敛域为$[-1,1]$

$3.(4)f(x)=\arcsin\dfrac{2x}{1+x^2}=\arctan\dfrac{2x}{1-x^2}=2\arctan x=\sum_{n=0}^{\infty}(-1)^n\dfrac{2}{2n+1}x^{2n+1}$

收敛域为$[-1,1]$

$$5.(1)\int_0^x e^{-t^2}dt=\int_0^x\sum_{n=0}^{\infty}\frac{(-t^2)^n}{n!}dt=\sum_{n=0}^{\infty}\int_0^x\frac{(-1)^n t^{2n}}{n!}dt=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{2n+1}}{2n+1}$$

收敛域为$\mathbb{R}$

$$5.(3)\int_0^x\frac{\sin t}{t}dt=\int_0^x\frac{\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}}{t}dt=\int_0^x\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n+1)!}dt$$

$$=\sum_{n=0}^{\infty}\int_0^x\frac{(-1)^n t^{2n}}{(2n+1)!}dt=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n)!}=x\cos x$$

收敛范围为$\mathbb{R}$

$7.(1)f(x)=\ln x$,按分式$\dfrac{x-1}{x+1}$的正整数幂次展开

设$u=\dfrac{x-1}{x+1}=1-\dfrac{2}{1+x}>-1(x>0)$,则$x=\dfrac{1+u}{1-u}$

于是$f(x)=\ln x=\ln\dfrac{1+u}{1-u}=\ln(1+u)-\ln(1-u)$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}u^{n+1}-\sum_{n=0}^{\infty}\frac{1}{n+1}u^{n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n-1}{n+1}\left(\frac{x-1}{x+1}\right)^{n+1}$$

$7.(3)f(x)=\dfrac{1}{1-x}$,按x的负整数次幂展开

设$u=\dfrac{1}{x}\neq 1(x\neq 1),x=\dfrac{1}{u}$

于是$f(x)=\dfrac{1}{1-\frac{1}{u}}=\dfrac{u}{u-1}=\dfrac{-u}{1-u}=-u\displaystyle\sum_{n=0}^{\infty}u^n=-\sum_{n=1}^{\infty}u^n=-\sum_{n=1}^{\infty}x^{-n}$

12.证明：幂级数的和函数在幂级数收敛域的内域(收敛域去掉端点的开区间)上是解析函数

Pf:记该内域为开区间I

幂级数的和函数$f(x)$在I上的幂级数收敛

对于任意$x_0\in I$,由于I是开区间,故存在$(\alpha,\beta)\subset I$,使得$x_0\in(\alpha,\beta)$

故$f(x)$可以在x_0处展开成幂级数,故它在x_0解析.

由于x_0是I中任意一个点，故$f(x)$在开区间I上解析.

13.设f和g是定义在开区间I上的两个解析函数. 证明以下两个条件中的

任何一个都蕴含着f和g在区间I上相等,即$f(x)=g(x),\forall x\in I$

(1)存在一点$x_0\in I$使得

$f^{(n)}(x_0)=g^{(n)}(x_0),\quad n=0,1,2\cdots$

(2)存在一列互不相同的点$x_n\in I,n=1,2,\cdots$,使得

$f(x_n)=g(x_n),\quad n=1,2,\cdots$

且$\{x_n\}_{n=1}^{\infty}$在I中有极限点

Pf: (1)不妨设$x_0=0$,考虑函数$h=f-g$,显然h也在I上解析

于是$h^{(n)}(0)=f^{(n)}(0)-g^{(n)}(0)=0,\quad n=0,1,2\cdots$

因为h在0处可以展开成幂级数,故存在$\delta>0$,使得对于$x\in(-\delta,\delta)\subseteq I$,有

$$h(x)=\sum_{n=0}^{\infty}\frac{h^{(n)}(0)}{n!}x^n\equiv 0$$

于是$f(x)=g(x),\forall x\in(-\delta,\delta)\subseteq I$

由定理12.3.5可知,$f(x)=g(x),\forall x\in I$

(2)设$h=f-g$,故$h(x_n)=f(x_n)-g(x_n)=0,n=1,2,\cdots.x_n\to x_0\in I\,(n\to\infty)$

不妨设$\{x_n\}_{n=1}^{\infty}$中任意两项不等,于是$h(x_n)-h(x_{n+1})=0-0=0$

$\Rightarrow\forall n,\ \exists x_{n,1}\in(x_n,x_{n+1})$,使得$h^{(1)}(x_{n,1})(x_n-x_{n+1})=0\Rightarrow h^{(1)}(x_{n,1})=0$

$\Rightarrow\forall n,\forall k$,有$h^{(k)}(x_{n,k})=0$

令$n\to\infty$,就有$h^{(k)}(x_0)=0$

由(1)可知：f和g在区间I上相等,即$f(x)=g(x),\forall x\in I$

1.(3) $f(x)=|\sin\frac{x}{2}|$ 是偶函数，故 $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=0 \quad \forall n\in\mathbb{N}$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}|\sin\frac{x}{2}|dx=\frac{2}{\pi}\int_0^{\pi}|\sin\frac{x}{2}|dx=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\sin x\,dx=\frac{4}{\pi}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|\sin\frac{x}{2}|\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}\sin\frac{x}{2}\cos nx\,dx=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\sin x\cos 2nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin(2n+1)x-\sin(2n-1)x\,dx=\frac{2}{\pi}\left(\frac{1}{2n+1}-\frac{1}{2n-1}\right)=-\frac{4}{\pi}\frac{1}{4n^2-1} \quad \forall n\in\mathbb{N}.$$

$$\Rightarrow f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{4n^2-1}$$

1.(5). $f(x)=\pi^2-x^2$（当 $|x|\le\pi$）是偶函数，故 $b_n=0 \quad \forall n\in\mathbb{N}$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2-x^2)dx=\frac{4}{3}\pi^2$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2-x^2)\cos nx\,dx=-\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\,dx=-\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx$$

$$=-\frac{2}{n\pi}\int_0^{\pi}x^2\,d\sin nx=\frac{2}{n\pi}\int_0^{\pi}\sin nx\,dx^2=\frac{4}{n\pi}\int_0^{\pi}x\sin nx\,dx$$

$$=\frac{-4}{n^2\pi}\int_0^{\pi}x\,d\cos nx=-\frac{4}{n^2\pi}\cdot\pi\cos n\pi+\frac{4}{n^2\pi}\int_0^{\pi}\cos nx\,dx=\frac{4}{n^2}\cdot(-1)^{n+1}$$

$$\Rightarrow f(x)\sim\frac{2}{3}\pi^2+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4\cos nx}{n^2}$$

1.(6) $f(x)=x^3 \ (-\pi<x\le\pi)$ 是奇函数. 于是 $a_n=0 \quad \forall n\in\mathbb{N}\cup\{0\}$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^3\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}x^3\sin nx\,dx=-\frac{2}{\pi}\frac{1}{n}\int_0^{\pi}x^3\,d\cos nx$$

$$=-\frac{2}{n\pi}\pi^3\cos n\pi+\frac{2}{n\pi}\int_0^{\pi}\cos nx\,dx^3=\frac{2\pi^2}{n}(-1)^{n+1}+\frac{6}{n\pi}\int_0^{\pi}x^2\cos nx\,dx$$

$$=\frac{2\pi^2}{n}(-1)^{n+1}+\frac{6}{n^2\pi}\int_0^{\pi}x^2\,d\sin nx=\frac{2\pi^2}{n}(-1)^{n+1}-\frac{6}{n^2\pi}\int_0^{\pi}\sin nx\,dx^2$$

$$=\frac{2\pi^2}{n}(-1)^{n+1}-\frac{12}{n^2\pi}\int_0^{\pi}x\sin nx\,dx=\frac{2\pi^2}{n}(-1)^{n+1}+\frac{12}{n^3\pi}\int_0^{\pi}x\,d\cos nx$$

$$=\frac{2\pi^2}{n}(-1)^{n+1}+\frac{12}{n^3\pi}\pi\cos n\pi-\frac{12}{n^3\pi}\int_0^{\pi}\cos nx\,dx=\frac{2\pi^2}{n}(-1)^{n+1}+\frac{12}{n^3}(-1)^n$$

$$f(x)\sim\sum_{n=1}^{\infty}\left[\frac{2\pi^2}{n}(-1)^{n+1}+\frac{12}{n^3}(-1)^n\right]\sin nx.$$

1.(8). $f(x) = \begin{cases} 0 & , -\pi \le x < 0 \text{ 时} \\ x & , 0 \le x < \pi \text{ 时} \end{cases}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx = \frac{1}{n\pi}\int_0^{\pi} x\,d\sin nx = -\frac{1}{n\pi}\int_0^{\pi}\sin nx\,dx$$

$$= -\frac{1}{n^2\pi}\int_0^{\pi} d\cos nx = -\frac{1}{n^2\pi}(\cos n\pi - 1) = \frac{1+(-1)^{n+1}}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx = -\frac{1}{n\pi}\int_0^{\pi} x\,d\cos nx$$

$$= -\frac{1}{n\pi}\cdot\pi\cos n\pi + \frac{1}{n\pi}\int_0^{\pi}\cos nx\,dx = \frac{(-1)^{n+1}}{n}$$

$$\Rightarrow f(x) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1+(-1)^{n+1}}{n^2\pi}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx.$$

2. 只需证: $n b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx$. ①

$$0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\,dx \quad ②$$

$$-n a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx \quad ③$$

①: $\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos nx\,df(x) = \frac{1}{\pi}\left[f(\pi)\cos n\pi - f(-\pi)\cos(-n\pi) - \int_{-\pi}^{\pi} f(x)\,d\cos nx\right]$

$$= \frac{1}{\pi}\cdot n\int_{-\pi}^{\pi} f(x)\sin nx\,dx = n b_n$$

②: $\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\,dx = \frac{1}{\pi}(f(\pi) - f(-\pi)) = 0$

③: $\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin nx\,df(x) = \frac{1}{\pi}\left[f(\pi)\sin(n\pi) - f(-\pi)\sin(-n\pi) - \int_{-\pi}^{\pi} f(x)\,d\sin nx\right]$

$$= -\frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = -n a_n \quad \square$$

4.(3) $f(x+\pi) = f(x) \Rightarrow a_{2n-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(2n-1)x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos(2n-1)x\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\cos(2n-1)x\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos(2n-1)x\,dx + \frac{1}{\pi}\int_{-\pi}^{0} f(x+\pi)\cos(2n-1)(x+\pi)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\left(\cos(2n-1)x - \cos(2n-1)x\right)dx = 0$$

$$b_{2n-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(2n-1)x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin(2n-1)x\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\sin(2n-1)x\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin(2n-1)x\,dx + \frac{1}{\pi}\int_{-\pi}^{0} f(x+\pi)\sin(2n-1)(x+\pi)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\left(\sin(2n-1)x - \sin(2n-1)x\right)dx = 0 \qquad \forall n \in \mathbb{N}. \quad \square$$

4.(5) $\begin{cases} f(x)=f(x+2\pi) \\ f(x)=f(\pi-x) \end{cases} \Rightarrow f(\pi-x)=f(x+2\pi) \Rightarrow x=\frac{3\pi}{2}$ 也是 $f(x)$ 对称轴.

$$a_{2n-1}=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(2n-1)x\,dx=\frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos(2n-1)x\,dx=\frac{1}{\pi}\int_0^{\pi} f(x)\cos(2n-1)x\,dx+\frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\cos(2n-1)x\,dx$$

$$\int_0^{\pi} f(x)\cos(2n-1)x\,dx=\int_0^{\pi} f(\pi-x)\cos(2n-1)(\pi-x)\,dx=-\int_0^{\pi} f(x)\cos(2n-1)x\,dx=0$$

$$\int_{\pi}^{2\pi} f(x)\cos(2n-1)x\,dx=\int_{\pi}^{2\pi} f(3\pi-x)\cos(2n-1)(3\pi-x)\,dx=-\int_{\pi}^{2\pi} f(x)\cos(2n-1)x\,dx=0$$

故. $a_{2n-1}=0 \quad \forall n\in\mathbb{N}$.

$$b_{2n}=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 2nx\,dx=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin 2nx\,dx=\frac{1}{\pi}\int_0^{\pi} f(x)\sin 2nx\,dx+\frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\sin 2nx\,dx$$

$$\int_0^{\pi} f(x)\sin 2nx\,dx=\int_0^{\pi} f(\pi-x)\sin\big((2n)(\pi-x)\big)\,dx=-\int_0^{\pi} f(x)\sin 2nx\,dx=0$$

$$\int_{\pi}^{2\pi} f(x)\sin 2nx\,dx=\int_{\pi}^{2\pi} f(3\pi-x)\sin\big(2n(3\pi-x)\big)\,dx=-\int_{\pi}^{2\pi} f(3\pi-x)\sin(2nx)\,dx=0$$

$\square$

故 $b_{2n}=0 \quad \forall n\in\mathbb{N}$.

6. (2) $\varphi(x)=\frac{1}{2}[f(x)+f(-x)]$ 是偶函数，故 $b_n'=0$. $\forall n\in\mathbb{N}$.

$$a_0'=\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(x)+f(-x)}{2}\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=a_0$$

$$a_n'=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(x)+f(-x)}{2}\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=a_n\quad\forall n\in\mathbb{N}.$$

6. (4) $h(x)=f(x+c)$.

$$a_0'=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+c)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=a_0$$

类似地，$a_n'=a_n$，$b_n'=b_n$，$\forall n\in\mathbb{N}$.

4. (1) 证: $x\in[-\pi,\pi]$ 时，$f'(x)=\frac{d\cos ax}{dx}=-a\sin ax$. $(a\notin\mathbb{Z},a>0)$

$\Rightarrow \forall x_0\in[-2\pi,2\pi]$，$|f(x_0+t)-f(x_0)|\le|f'(\xi)|\,|x_0+t-x_0|\le a|t|$

故 f 在 ~~[-π,π]~~ $(-2\pi,2\pi)$ Lipschitz 连续

由于 f 周期为 2π，故 f 在 $\mathbb{R}$ 上 Lipschitz 连续

$\Rightarrow \forall x\in\mathbb{R}$. f 在 x 处的 Fourier 级数收敛到 $f(x)$

$$\Rightarrow f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos ax\,dx=\frac{2}{\pi}\int_0^{\pi}\cos ax\,dx=\frac{2}{\pi}\frac{1}{a}\int_0^{\pi}d\sin ax=\frac{2\sin a\pi}{a\pi}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos ax\cos nx\,dx=\cancel{\frac{1}{2\pi}\int_{-\pi}}$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(a+n)x+\cos(n-a)x\,dx=\frac{1}{\pi}\int_0^{\pi}\cos(a+n)x+\cos(n-a)x\,dx$$

$$\cancel{=\frac{1}{\pi}\left(\frac{1}{a+n}+\frac{1}{n-a}\right)-\frac{1}{\pi}\cdot\frac{2n}{n^2-a^2}}=\frac{1}{\pi}\frac{\sin(a+n)\pi}{a+n}+\frac{1}{\pi}\frac{\sin(n-a)\pi}{n-a}=(-1)^{n-1}\frac{2a\sin a\pi}{(n^2-a^2)\pi}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos ax\sin nx\,dx=0\qquad\forall n\in\mathbb{N}.$$

$$\Rightarrow f(x)=\frac{\sin a\pi}{a\pi}+\frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{a\cos nx}{n^2-a^2}$$

4.(2). 由(1)可知. $\cos ax = \frac{\sin a\pi}{a\pi} + \frac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos nx}{n^2-a^2}$, $\forall x\in\mathbb{R}$. — (*)

令 $x=\pi$, 就有: $\cos a\pi = \frac{\sin a\pi}{a\pi} + \frac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos n\pi}{n^2-a^2}$

其中: $\cos n\pi = (-1)^n$. 于是 $\sum_{n=1}^{\infty}\frac{1}{n^2-a^2} = \frac{1}{2a^2} - \frac{\pi}{2a}\cot a\pi$.

4.(3). 在(*)中令 $x=$ ~~2π~~ 0. (注意: 只能在$[-\pi,\pi]$内取x, 因为$f(x)$是$\cos ax$, $x\in[-\pi,\pi]$ 的2π周期延拓!) 于是 $1 = \frac{\sin a\pi}{a\pi} + \frac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2-a^2}$

$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-a^2} = -\frac{1}{2a^2} + \frac{\pi}{2a\sin a\pi}$. —— (**)

4.(4). 在(**)中令 $a=\frac{1}{2}$. 于是 LHS $= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-\frac{1}{4}} = 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}$

RHS $= -2+\pi \Rightarrow \frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1} = \frac{\pi}{4}$

□

5.(1). $f(x)$在$(-\pi,\pi]$上等于$\sin ax$. 注意到f在$(2k-1)\pi$. $k\in\mathbb{Z}$处间断

(因为 $\lim_{x\to\pi^+}f(x) = \lim_{x\to-\pi^+}f(x) = -\sin a\pi$. $\lim_{x\to\pi^-}f(x) = \lim_{x\to\pi^-}\sin ax = \sin a\pi$,

又有 $a\in\mathbb{R}^+\setminus\mathbb{Z}$. f是$\sin ax$以2π为周期的周期延拓)

由于 $f'(x) = \frac{d\sin ax}{dx} = a\cos ax \in[-a,a]$. $\forall x\neq(2m-1)\pi$. $m\in\mathbb{Z}$.

于是 f在任何不包含$\{(2m-1)\pi \mid m\in\mathbb{Z}\}$中点的区间上Lipschitz连续

$\Rightarrow f$在$\mathbb{R}\setminus\{(2m-1)\pi\mid m\in\mathbb{Z}\}$处的Fourier级数收敛.

即 $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}a_n\cos nx + b_n\sin nx$. $\forall x\in\mathbb{R}\setminus\{(2m-1)\pi\mid m\in\mathbb{Z}\}$.

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin ax\,dx = 0$. $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin ax\cos nx\,dx = 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin ax\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin ax\sin nx\,dx$

$= \frac{1}{\pi}\int_{0}^{\pi}\cos(a+n)x - \cos(n-a)x\,dx = \frac{1}{\pi}\cdot\frac{2n\sin a\pi}{n^2-a^2}(-1)^{n-1}$

$\Rightarrow f(x) = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n\sin nx}{n^2-a^2}$. $\forall x\in\mathbb{R}\setminus\{(2m-1)\pi\mid m\in\mathbb{Z}\}$

5. (2). 由(1)可知：$\sin ax = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nx}{n^2-a^2} \quad \forall x\in\mathbb{R}\setminus\{(2m-1)\pi \mid m\in\mathbb{Z}\}$ ⋯ (*)

在(*)中令 $x=\frac{\pi}{2}$：$\sin\frac{a\pi}{2} = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin\frac{n\pi}{2}}{n^2-a^2}$ ⋯ (**)

n	1	2	3	4	5	⋯
$\sin\frac{n\pi}{2}$	1	0	-1	0	1	⋯

于是 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin\frac{n\pi}{2}}{n^2-a^2} = (-1)^0\frac{1}{1^2-a^2}+0+(-1)^2\frac{-1}{3^2-a^2}+0$

$+(-1)^4\frac{1}{5^2-a^2}+0+(-1)^6\frac{-1}{7^2-a^2}+0+\cdots$

观察可得上式 $=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(2n-1)^2-a^2}$

再由(**)可知：$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(2n-1)^2-a^2} = \frac{\pi\sin\frac{a\pi}{2}}{2\sin a\pi}$，其中 $a\in\mathbb{R}^+\setminus\mathbb{Z}$ □

8. (1) $F(x)=\int_0^x f(t)\,dt - \frac{1}{2}x\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\,dt$

$F(x+2\pi)-F(x) = \int_x^{x+2\pi}f(t)\,dt - \frac{1}{2}\cdot 2\pi\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\,dt$

(由f周期性) $= \int_0^{2\pi}f(t)\,dt - \int_{-\pi}^{\pi}f(t)\,dt = 0 \quad \forall x\in\mathbb{R}.$

于是 $F(x)$ 以 2π 为周期.

Check：$F\in C(\mathbb{R})$. $\cancel{|F(x+h)-F(x)|}$

$\forall x\in\mathbb{R}$, $|F(x+h)-F(x)| = \left|\int_x^{x+h}f(t)\,dt+\frac{a_0}{2}h\right| \le \left|\int_x^{x+h}f(t)\,dt\right|+\left|\frac{a_0}{2}h\right| \to 0 \ (\text{as } h\to 0)$

$\therefore F\in C(\mathbb{R})$

[注：虽然 ~~F在R上~~ F是 $f(x)-\frac{a_0}{2}$ 在R上的原函数之一. 但F不一定可微]

于是在有界闭区间 $[-\pi,\pi]$ 上. 存在 $M>0$. 使得 $|f(x)|<M$. $\forall x\in[-\pi,\pi]$

由f周期性：$|f(x)|<M \ \forall x\in\mathbb{R}$.

于是 $\forall x_0\in\mathbb{R}$. $|F(x_0+t)-F(x_0)| = \left|\int_{x_0}^{x_0+t}f(s)\,ds+\frac{a_0}{2}t\right|$

$\le M\left|\int_{x_0}^{x_0+t}ds\right|+\left|\frac{a_0}{2}\right||t| = \left(M+\left|\frac{a_0}{2}\right|\right)|s|$ 故 $F(x)$ 在 $\mathbb{R}$ 上 Lipschitz 连续.

8.(2) (i). 若 $f\in C(\mathbb{R})$. 则 $F\in D(\mathbb{R})$. $F'(x)=f(x)-\frac{a_0}{2}$. $\forall x\in\mathbb{R}$.

由于 F 在 $\mathbb{R}$ 上 Lipschitz 连续. 故 F 的 Fourier 级数在 $\mathbb{R}$ 上收敛到 F.

$$F(x)=\frac{a_0'}{2}+\sum_{n=1}^{\infty}(a_n'\cos nx+b_n'\sin nx)$$

一方面. $\frac{1}{\pi}\int_{-\pi}^{\pi}F'(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos nx\,dF(x)$

$$=\frac{1}{\pi}\left(F(\pi)\cos n\pi-F(-\pi)\cos(-\pi n)\right)-\frac{1}{\pi}\int_{-\pi}^{\pi}F(x)\,d\cos nx$$

$$=\frac{n}{\pi}\int_{-\pi}^{\pi}F(x)\sin nx\,dx=nb_n'$$

另一方面. $\frac{1}{\pi}\int_{-\pi}^{\pi}F'(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x)-\frac{a_0}{2}\right)\cos nx\,dx$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=a_n$$

$$\Rightarrow\quad b_n'=\frac{a_n}{n}$$

类似地，可知：$a_n'=-\frac{b_n}{n}$

于是此时 $F(x)$ 的 Fourier 级数为 $\frac{a_0'}{2}+\sum_{n=1}^{\infty}\left[\left(-\frac{b_n}{n}\right)\cos nx+\frac{a_n}{n}\sin nx\right]$ （求不出 a_0'）

(ii). 若 f 在 $\mathbb{R}$ 上不连续. 结合 f 在 $\mathbb{R}$ 上 Riemann 可积. ~~由 Stone - Weierstrass 定理可知.~~ 由于有界闭区间 $[-\pi,\pi]$ 上的 Riemann 可积函数可以由分段线性函数一致逼近. 故 $\exists$ 一列 $\{f_k\}_{k=1}^{\infty}$ 在 $[-\pi,\pi)$ 上. 使得

$$\{f_k\}_{k=1}^{\infty}\rightrightarrows f\quad\text{on }[-\pi,\pi]$$

~~这 f_k 在 $[-\pi,\pi]$ 上连续.~~

对这些 f_k 进行 2π 周期延拓. 到 $\mathbb{R}$ 上. $(k=1,2,\dots)$

于是 $f_k(x) \sim \frac{a_{0,k}}{2} + \sum_{n=1}^{\infty}(a_{n,k}\cos nx + b_{n,k}\sin nx)$, $\forall k$

记 $F_k(x) = \int_0^x f_k(t)dt - \frac{a_{0,k}}{2}x$

由于 $\{f_k\}_{k=1}^{\infty}$ 在 $\mathbb{R}$ 上一致 ~~收敛~~ 收敛于 f. 故 $\{f_k\}_{k=1}^{\infty}$ 在 $[-\pi,\pi]$ 上积分平均收敛于 f. 即

$$\int_{-\pi}^{\pi}|f_k(x)-f(x)|dx \to 0 \quad (k\to\infty)$$

于是 $|F(x) - F_k(x)| = \left|\int_0^x f(t)dt - \int_0^x f_k(t)dt - \frac{a_0}{2}x + \frac{a_{0,k}}{2}x\right|$

$$\le \int_0^x |f(t)-f_k(t)|dt + \left|\frac{a_0}{2}-\frac{a_{0,k}}{2}\right||x| \quad \forall x\in[-\pi,\pi]$$

$$\le \int_{-\pi}^{\pi}|f(t)-f_k(t)|dt + \frac{|x|}{2}|a_0-a_{0,k}| \to 0 \quad (k\to\infty)$$

于是 $\lim_{k\to\infty}F_k(x) = F(x)$, $\forall x\in[-\pi,\pi]$. 由周期性: 可知 $\lim_{k\to\infty}F_k(x)=F(x)$, $\forall x\in\mathbb{R}$

由 8.(1) 和 8.(2)(i) 可知: $F_k(x) = \frac{a_0'}{2} + \sum_{n=1}^{\infty}\left[\left(-\frac{b_{n,k}}{n}\right)\cos nx + \frac{a_{n,k}}{n}\sin nx\right]$

于是 $F(x) = \lim_{k\to\infty}F_k(x) = \frac{a_0'}{2} + \sum_{n=1}^{\infty}\left[\left(-\frac{b_n}{n}\right)\cos nx + \frac{a_n}{n}\sin nx\right]$ □

8.(3) 由于 $F(x) = \frac{a_0'}{2} + \sum_{n=1}^{\infty}\left[\left(-\frac{b_n}{n}\right)\cos nx + \frac{a_n}{n}\sin nx\right]$, $\forall x\in[-\pi,\pi]$ ...... (*)

令 $x=0$, 则 $F(0) = \frac{a_0'}{2} + \sum_{n=1}^{\infty}\left(-\frac{b_n}{n}\right)$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{b_n}{n} = \frac{a_0'}{2} - F(0) \quad \text{其中 } F(0) = \int_0^0 f(t)dt - \frac{a_0}{2}\cdot 0 = 0$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{b_n}{n} = \frac{a_0'}{2}$$ ~~是某个由~~

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_0^x f(t)dt - \frac{a_0}{2}x\right)dx$$

由 f.

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^x f(t)dt\,dx$$ 是一个确定的常数

故级数 $\sum_{n=1}^{\infty}\frac{b_n}{n}$ 收敛.

由 (*) 可知: $\int_0^x f(t)dt = \frac{a_0}{2}x + \frac{a_0'}{2} + \sum_{n=1}^{\infty}\left(-\frac{b_n}{n}\cos nx + \frac{a_n}{n}\sin nx\right)$

$$= \frac{a_0}{2}x + \sum_{n=1}^{\infty}\frac{b_n}{n} + \sum_{n=1}^{\infty}\left(-\frac{b_n}{n}\cos nx + \frac{a_n}{n}\sin nx\right)$$

$$= \int_0^x \frac{a_0}{2}dt + \sum_{n=1}^{\infty}\int_0^x (a_n\cos nt + b_n\sin nt)dt.$$ □