10/23 作业

$$1.proof: 1 \le \lim_{x\to\infty}\sqrt{1+\frac{1}{x^2}} \le \lim_{x\to\infty}\left(1+\frac{1}{x^2}\right) = 1+\lim_{x\to\infty}\frac{1}{x^2} \le 1+\lim_{x\to\infty}\frac{1}{x} \overset{Heine\ Thm}{=} 1+\lim_{n\to\infty}\frac{1}{n}$$

$$\forall\varepsilon>0, choose\ N=\left[\frac{1}{\varepsilon}\right]+1\in\mathbb{N}, s.t.\forall n>N, \frac{1}{n}<\frac{1}{\left[\frac{1}{\varepsilon}\right]+1}<\varepsilon \Rightarrow \lim_{n\to\infty}\frac{1}{n}=0 \Rightarrow \lim_{x\to\infty}\sqrt{1+\frac{1}{x^2}}=1$$

$$2.(2)\lim_{x\to1^-}\frac{\sqrt[3]{1-x}-\sqrt[4]{1-x}}{\sqrt[3]{1-x}+3\sqrt[4]{1-x}} = \lim_{x\to1^-}\frac{\sqrt[12]{1-x}-1}{\sqrt[12]{1-x}+3} = \frac{\lim_{x\to1^-}\sqrt[12]{1-x}-1}{\lim_{x\to1^-}\sqrt[12]{1-x}+3} = -\frac{1}{3}$$

$$2.(4)\lim_{x\to1^-}\frac{[3x]}{x+2} = \lim_{\varepsilon\to0^+}\frac{[3(1-\varepsilon)]}{3-\varepsilon} = \frac{2}{3}$$

$$2.(6)\lim_{x\to2^-}\frac{[x]^2-1}{x^2-1} = \lim_{\varepsilon\to0^+}\frac{[(2-\varepsilon)]^2-1}{(2-\varepsilon)^2-1} = \lim_{\varepsilon\to0^+}\frac{[(2-\varepsilon)]^2-1}{\varepsilon^2-4\varepsilon+3} = 0$$

$$2.(8)\lim_{x\to2^-}\arctan\frac{\sqrt{x-1}}{x-2} = \arctan\lim_{x\to2^-}\frac{\sqrt{x-1}}{x-2} = \arctan\lim_{y\to-\infty}y = \lim_{y\to-\infty}\arctan y = -\frac{\pi}{2}$$

$$2.(10)\lim_{x\to0^-}\frac{1}{1+2^{\frac{1}{x}}} = \lim_{x\to-\infty}\frac{1}{1+2^x} = \frac{1}{1+\lim_{x\to-\infty}2^x} = 1$$

$$3.(1)\lim_{x\to+\infty}\left(\sqrt{(x+a)(x+b)}-x\right) = \lim_{x\to+\infty}\frac{(x+a)(x+b)-x^2}{\sqrt{(x+a)(x+b)}+x} = \lim_{x\to+\infty}\frac{(a+b)x+ab}{\sqrt{(x+a)(x+b)}+x} = \lim_{x\to+\infty}\frac{(a+b)+\frac{ab}{x}}{\sqrt{\left(1+\frac{a}{x}\right)\left(1+\frac{b}{x}\right)}+1} = \frac{a+b}{2}$$

$$3.(4)\lim_{x\to+\infty}\frac{\left(x+\sqrt{x^2-2x}\right)^n+\left(x-\sqrt{x^2-2x}\right)^n}{\sqrt[3]{x^{3n}+1}+\sqrt[3]{x^{3n}-1}} = \lim_{x\to+\infty}\frac{\left(1+\sqrt{1-\frac{2}{x}}\right)^n+\left(1-\sqrt{1-\frac{2}{x}}\right)^n}{\sqrt[3]{1+\frac{1}{x^{3n}}}+\sqrt[3]{1-\frac{1}{x^{3n}}}} = \frac{2^n}{2} = 2^{n-1}$$

$$3.(5)\lim_{x\to\infty}x^{\frac{1}{3}}\left[(x+1)^{\frac{2}{3}}-(x-1)^{\frac{2}{3}}\right] = \lim_{x\to\infty}x^{\frac{1}{3}}\frac{(x+1)^2-(x-1)^2}{(x+1)^{\frac{4}{3}}-(x-1)^{\frac{2}{3}}(x+1)^{\frac{2}{3}}+(x-1)^{\frac{4}{3}}}$$

$$=\lim_{x\to\infty}\frac{4x}{(x+1)\left(1+\frac{1}{x}\right)^{\frac{1}{3}}-\left(x^2-1\right)^{\frac{2}{3}}\cdot\frac{1}{x^{\frac{1}{3}}}+(x-1)\left(1-\frac{1}{x}\right)^{\frac{1}{3}}}$$

$$=\lim_{x\to\infty}\frac{4}{\left(1+\frac{1}{x}\right)^{\frac{4}{3}}-\left(1-\frac{1}{x^2}\right)^{\frac{2}{3}}+\left(1-\frac{1}{x}\right)^{\frac{4}{3}}} = 4$$

$$3.(7)\ 0\le \lim_{x\to+\infty}\left|\sin\sqrt{x+1}-\sin\sqrt{x}\right|=\lim_{x\to+\infty}2\left|\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}\right|$$

$$\le\lim_{x\to+\infty}2\left|\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\right|=\lim_{x\to+\infty}2\left|\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\right|=\lim_{x\to+\infty}2\left|\sin\frac{1}{2\left(\sqrt{x+1}+\sqrt{x}\right)}\right|=0$$

$$\Rightarrow\lim_{x\to+\infty}\left|\sin\sqrt{x+1}-\sin\sqrt{x}\right|=0$$

$$4.(3)\lim_{x\to0}\frac{\sin3x-\sin2x}{\sin5x}=\lim_{x\to0}\frac{\frac{\sin3x}{x}-\frac{\sin2x}{x}}{\frac{\sin5x}{x}}=\frac{1}{5}$$

$$4.(6)\lim_{x\to0}\frac{\cos3x-\cos2x}{\sin x^2}=\lim_{x\to0}\frac{2\sin\frac{x}{2}\sin\frac{5x}{2}}{\sin x^2}=\lim_{x\to0}\frac{2\frac{\sin\frac{x}{2}}{x}\frac{\sin\frac{5x}{2}}{x}}{\frac{\sin x^2}{x^2}}=\frac{5}{2}$$

$$4.(7)\lim_{x\to\frac{\pi}{4}}\tan2x\tan\left(\frac{\pi}{4}-x\right)=\lim_{x\to\frac{\pi}{4}}\frac{2\tan x}{1-\tan^2x}\frac{1-\tan x}{1+\tan x}=\lim_{x\to\frac{\pi}{4}}\frac{2\tan x}{(1+\tan x)^2}=\frac{1}{2}$$

$$4.(10)\mathit{lemma}:\lim_{x\to0^+}\frac{1-\cos x}{x^2}=\lim_{x\to0^+}\frac{1}{4}\frac{2\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}=\frac{1}{2}\lim_{x\to0^+}\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}=$$

$$\lim_{x\to0^+}\frac{1-\sqrt{\cos x}}{\left(1-\cos\sqrt{x}\right)^2}=\lim_{x\to0^+}\frac{\frac{1-\sqrt{\cos x}}{x^2}}{\left(\frac{1-\cos\sqrt{x}}{x}\right)^2}=\lim_{x\to0^+}\frac{\frac{1-\cos x}{x^2}\frac{1}{1+\sqrt{\cos x}}}{\left(\frac{1-\cos\sqrt{x}}{x}\right)^2}=\frac{1}{2}\lim_{x\to0^+}\frac{\frac{1-\cos x}{x^2}}{\left(\frac{1-\cos\sqrt{x}}{\left(\sqrt{x}\right)^2}\right)^2}=\frac{1}{2}\frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2}=1$$

$$5.(2)e^2=\lim_{x\to0^+}\left[(1+2x)^{\frac{1}{2x}}\right]^2\le\lim_{x\to0^+}\left(e^x+x\right)^{\frac{1}{x}}=\lim_{x\to0^+}\left(1+\frac{x}{e^x}\right)^{\frac{1}{x}}e\le e\lim_{x\to0^+}\left(1+\frac{x}{e^x}\right)^{\frac{e^x}{x}}=e^2\Rightarrow\lim_{x\to0^+}\left(e^x+x\right)^{\frac{1}{x}}=e^2$$

$$e^2=\lim_{x\to0^-}\left[(1+2x)^{\frac{1}{2x}}\right]^2\le\lim_{x\to0^-}\left(e^x+x\right)^{\frac{1}{x}}=\lim_{x\to0^-}\left(1+\frac{x}{e^x}\right)^{\frac{1}{x}}e\le e\lim_{x\to0^-}\left(1+\frac{x}{e^x}\right)^{\frac{e^x}{x}}=e^2\Rightarrow\lim_{x\to0^-}\left(e^x+x\right)^{\frac{1}{x}}=e^2$$

$$\Rightarrow\lim_{x\to0}\left(e^x+x\right)^{\frac{1}{x}}=e^2$$

$$5.(4)\mathit{lemma}:1=\ln\left(\lim_{x\to0}(1+x)^{\frac{1}{x}}\right)=\lim_{x\to0}\ln(1+x)^{\frac{1}{x}}=\lim_{x\to0}\frac{\ln(1+x)}{x}=\lim_{x\to\infty}x\ln\left(1+\frac{1}{x}\right)$$

$$\lim_{x\to\infty}\left(\frac{x^2+1}{x^2-1}\right)^x=\lim_{x\to\infty}\left(1+\frac{2}{x^2-1}\right)^x=e^{\lim_{x\to\infty}x\ln\left(1+\frac{2}{x^2-1}\right)}=e^{\lim_{x\to\infty}\frac{x}{x^2-1}\lim_{x\to\infty}\left(x^2-1\right)\ln\left(1+\frac{2}{x^2-1}\right)}=e^{0\cdot e}=1$$

$$5.(5)\lim_{x\to0}\frac{\ln\left(x^2+e^x\right)}{\ln\left(x^3+e^{2x}\right)}=\lim_{x\to0}\frac{\frac{1}{x}\ln\left(x^2+e^x\right)}{\frac{1}{x}\ln\left(x^3+e^{2x}\right)}=\frac{\lim_{x\to0}\frac{1}{x}\ln\left(x^2+e^x\right)}{\lim_{x\to0}\frac{1}{x}\ln\left(x^3+e^{2x}\right)}$$

$$\lim_{x\to0}\frac{1}{x}\ln\left(x^2+e^x\right)=\lim_{x\to0}\frac{1}{x}\ln\left(1+\frac{x^2}{e^x}\right)+1=\lim_{x\to0}\frac{1}{x}\frac{x^2}{e^x}\frac{\ln\left(1+\frac{x^2}{e^x}\right)}{\frac{x^2}{e^x}}+1=\lim_{x\to0}\frac{x}{e^x}\lim_{x\to0}\frac{\ln\left(1+\frac{x^2}{e^x}\right)}{\frac{x^2}{e^x}}+1=1$$

$$\lim_{x\to0}\frac{1}{x}\ln\left(x^3+e^{2x}\right)=\lim_{x\to0}\frac{1}{x}\ln\left(1+\frac{x^3}{e^{2x}}\right)+2=\lim_{x\to0}\frac{1}{x}\frac{x^3}{e^{2x}}\frac{\ln\left(1+\frac{x^3}{e^{2x}}\right)}{\frac{x^3}{e^{2x}}}+2=\lim_{x\to0}\frac{x^2}{e^{2x}}\lim_{x\to0}\frac{\ln\left(1+\frac{x^3}{e^{2x}}\right)}{\frac{x^3}{e^{2x}}}+2=2$$

$$\Rightarrow\lim_{x\to0}\frac{\ln\left(x^2+e^x\right)}{\ln\left(x^3+e^{2x}\right)}=\frac{1}{2}$$

$$5.(10)\ \lim_{x\to+\infty}\frac{\ln\left(1+3^x\right)}{\ln\left(1+2^x\right)}=\lim_{x\to+\infty}\frac{\frac{1}{x}\ln\left(1+3^x\right)}{\frac{1}{x}\ln\left(1+2^x\right)}=\frac{\lim\limits_{x\to+\infty}\frac{1}{x}\ln\left(1+3^x\right)}{\lim\limits_{x\to+\infty}\frac{1}{x}\ln\left(1+2^x\right)}$$

$$\lim_{x\to+\infty}\frac{1}{x}\ln\left(1+3^x\right)=\lim_{x\to+\infty}\left\{\frac{1}{x}\left[\ln\left(1+3^x\right)-\ln 3^x\right]+\frac{1}{x}\ln 3^x\right\}=\lim_{x\to+\infty}\frac{\ln\left(1+3^{-x}\right)}{x}+\ln 3=\ln 3$$

$$\lim_{x\to+\infty}\frac{1}{x}\ln\left(1+2^x\right)=\lim_{x\to+\infty}\left\{\frac{1}{x}\left[\ln\left(1+2^x\right)-\ln 2^x\right]+\frac{1}{x}\ln 2^x\right\}=\lim_{x\to+\infty}\frac{\ln\left(1+2^{-x}\right)}{x}+\ln 2=\ln 2$$

$$\Rightarrow\lim_{x\to+\infty}\frac{\ln\left(1+3^x\right)}{\ln\left(1+2^x\right)}=\frac{\ln 3}{\ln 2}$$

$$6.(3)\lim_{x\to 0}\left(\frac{\cos x}{\cos 2x}\right)^{\frac{1}{x^2}}=e^{\lim\limits_{x\to 0}\frac{1}{x^2}\ln\left(\frac{\cos x}{\cos 2x}\right)}=e^{\lim\limits_{x\to 0}\frac{1}{x^2}\frac{\cos x-\cos 2x}{\cos 2x}\lim\limits_{x\to 0}\frac{\ln\left(1+\frac{\cos x-\cos 2x}{\cos 2x}\right)}{\frac{\cos x-\cos 2x}{\cos 2x}}}=e^{\lim\limits_{x\to 0}\frac{\frac{\cos x-\cos 2x}{x^2}}{\cos 2x}}=e^{\frac{\lim\limits_{x\to 0}\frac{\cos x-\cos 2x}{x^2}}{\lim\limits_{x\to 0}\cos 2x}}$$

$$=e^{\lim\limits_{x\to 0}\frac{\cos x-\cos 2x}{x^2}}=e^{-\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}+4\lim\limits_{x\to 0}\frac{1-\cos 2x}{(2x)^2}}=e^{\frac{3}{2}}$$

$$6.(4)\lim_{x\to 0}\left(\cos\sqrt{x}\right)^{\frac{1}{x}}=e^{\lim\limits_{x\to 0}\frac{1}{x}\ln\cos\sqrt{x}}=e^{\lim\limits_{x\to 0}\frac{1}{x}\ln\left[1+\left(\cos\sqrt{x}-1\right)\right]}=e^{\lim\limits_{x\to 0}\frac{\cos\sqrt{x}-1}{\left(\sqrt{x}\right)^2}\lim\limits_{x\to 0}\frac{\ln\left[1+\left(\cos\sqrt{x}-1\right)\right]}{\left(\cos\sqrt{x}-1\right)}}=e^{-\frac{1}{2}}$$

$$6.(5)\lim_{x\to\frac{\pi}{4}}\left(\tan x\right)^{\tan 2x}=\lim_{x\to\frac{\pi}{4}}\left(\tan x\right)^{\frac{2\tan x}{1-\tan^2 x}}=\lim_{x\to 1}x^{\frac{2x}{1-x^2}}=e^{\lim\limits_{x\to 1}\frac{2x}{1-x^2}\ln x}=e^{\lim\limits_{x\to 1}\frac{2x}{1-x^2}\ln\left[1+(x-1)\right]}=e^{\lim\limits_{x\to 1}\frac{2x}{1-x^2}(x-1)\lim\limits_{x\to 1}\frac{\ln\left[1+(x-1)\right]}{(x-1)}}$$

$$=e^{-\lim\limits_{x\to 1}\frac{2x}{x+1}\lim\limits_{x\to 1}\frac{\ln\left[1+(x-1)\right]}{(x-1)}}=e^{-1}$$

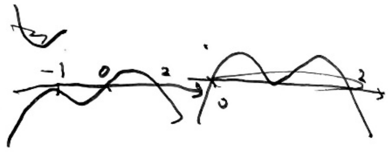
$$6.(8)\lim_{x\to 0}\frac{\ln\cos ax}{\ln\cos bx}=\frac{\lim\limits_{x\to 0}\frac{1}{x^2}\ln\cos ax}{\lim\limits_{x\to 0}\frac{1}{x^2}\ln\cos bx}$$

$$\lim_{x\to 0}\frac{1}{x^2}\ln\cos ax=\lim_{x\to 0}\frac{1}{x^2}\ln\left[1+\left(\cos ax-1\right)\right]=\lim_{x\to 0}\frac{\cos ax-1}{x^2}\lim_{x\to 0}\frac{\ln\left[1+\left(\cos ax-1\right)\right]}{\cos ax-1}=a^2\lim_{x\to 0}\frac{\cos ax-1}{\left(ax\right)^2}=-\frac{1}{2}a^2$$

$$\lim_{x\to 0}\frac{1}{x^2}\ln\cos bx=\lim_{x\to 0}\frac{1}{x^2}\ln\left[1+\left(\cos bx-1\right)\right]=\lim_{x\to 0}\frac{\cos bx-1}{x^2}\lim_{x\to 0}\frac{\ln\left[1+\left(\cos bx-1\right)\right]}{\cos bx-1}=b^2\lim_{x\to 0}\frac{\cos bx-1}{\left(bx\right)^2}=-\frac{1}{2}b^2$$

$$\Rightarrow\lim_{x\to 0}\frac{\ln\cos ax}{\ln\cos bx}=\frac{a^2}{b^2}$$

1、(1)、要证：$y=\sqrt{x^2+1}$ 在 $\mathbb{R}$ 上连续



~~$\forall x_0 \in \mathbb{R}$,~~

对于一个给定的 $x_0 \in \mathbb{R}$，$\forall \varepsilon>0$，取 $\delta=\varepsilon>0$，则有 $\forall\, 0<|x-x_0|<\delta$.

$$\left|\sqrt{x^2+1}-\sqrt{x_0^2+1}\right| = \frac{|x+x_0||x-x_0|}{\sqrt{x^2+1}+\sqrt{x_0^2+1}} \leq \frac{(|x|+|x_0|)|x-x_0|}{\sqrt{x^2+1}+\sqrt{x_0^2+1}} < \frac{(\sqrt{x^2+1}+\sqrt{x_0^2+1})|x-x_0|}{\sqrt{x^2+1}+\sqrt{x_0^2+1}} = \varepsilon$$

$\Rightarrow$ $y=\sqrt{x^2+1}$ 在 $\mathbb{R}$ 上连续

1、(2)、对于一个给定的 $x_0 \in \mathbb{R}$，$\forall \varepsilon>0$，取 $\delta=\min\left\{\frac{\varepsilon}{2|3x_0^2+3|x_0|+1|}, 1\right\}>0$ 则有 $\forall\, 0<|x-x_0|<\delta$.

$$|\sin(2x^3+1)-\sin(2x_0^3+1)| = |2\sin(x^3-x_0^3)\cos(x^3+x_0^3+1)| \leq |2\sin(x^3-x_0^3)|$$

$$< 2|x^3-x_0^3| = 2|x-x_0||x^2+xx_0+x_0^2| \leq 2|x-x_0|\,|x^2+|xx_0|+x_0^2|$$

$$= 2|x-x_0|\,|(|x_0|+\delta)^2+(|x_0|+\delta)|x_0|+x_0^2| = 2\delta|3x_0^2+3\delta|x_0|+\delta^2|$$

$$\leq 2\delta|3x_0^2+3|x_0|+1| \leq \varepsilon.$$

$\Rightarrow$ $y=\sin(2x^3+1)$ 在 $\mathbb{R}$ 上连续

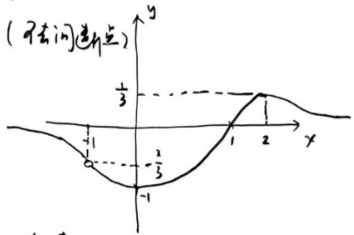
2.(2). $y=\frac{x^2-1}{x^3+1}$ 在 $x=-1$ 处不连续

$\lim\limits_{x\to(-1)^-}\frac{x^2-1}{x^3+1}=\lim\limits_{x\to(-1)^-}\frac{(x-1)(x+1)}{(x+1)(x^2-x+1)}=\lim\limits_{x\to(-1)^-}\frac{x-1}{x^2-x+1}=-\frac{2}{3}$

$\lim\limits_{x\to(-1)^+}\frac{x^2-1}{x^3+1}=\lim\limits_{x\to(-1)^+}\frac{x-1}{x^2-x+1}=-\frac{2}{3}$

⇒ 间断点是第一类间断点.(可去间断点)

如图:

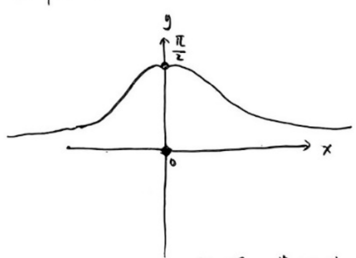


2.(4). $y=\frac{|x|}{x}\arctan\frac{1}{x}$. 在 $x=0$ 处不连续

$\lim\limits_{x\to0^-}\frac{|x|}{x}\arctan\frac{1}{x}=\lim\limits_{x\to0^+}\frac{|-x|}{-x}\arctan\frac{1}{-x}=\lim\limits_{x\to0^+}\frac{|x|}{x}\arctan\frac{1}{x}$

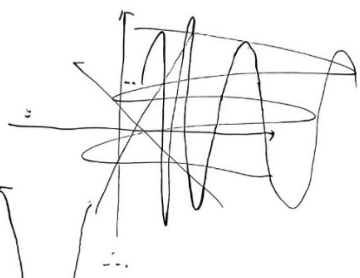
⇒ 间断点是第一类间断点(

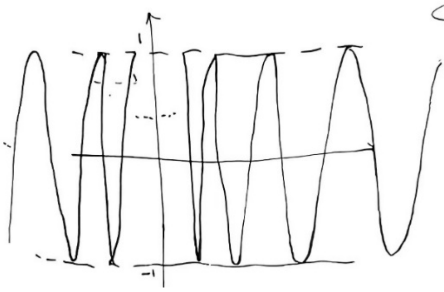
如图:



2.(6). $y=\sin\frac{1}{x}$ 在 $x=0$ 处不连续、间断点是第二类间断点(振荡间断点).

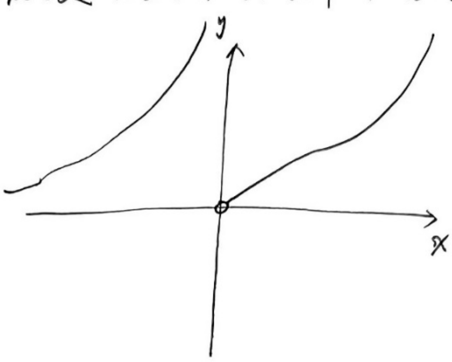
如图:





(8). $y=e^{x-\frac{1}{x}}$ 不连续点在 $x=0$ 处 为第二类间断点(无穷间断点)

如图:



$$4.(2)\, m(x)=\min\{f(x),g(x)\}=\frac{f(x)+g(x)-|f(x)-g(x)|}{2}$$

for any given $x_0 \in I, \forall \varepsilon > 0,$

$\exists \delta_1 > 0, s.t. \forall x \in I : 0 < |x-x_0| < \delta_1,$

$|f(x)-f(x_0)| < \frac{\varepsilon}{2}$

$\exists \delta_2 > 0, s.t. \forall x \in I : 0 < |x-x_0| < \delta_2,$

$|g(x)-g(x_0)| < \frac{\varepsilon}{2}$

choose $\delta = \min\{\delta_1,\delta_2\} > 0,$ *then* $\forall x \in I : 0 < |x-x_0| < \delta,$

$$|m(x)-m(x_0)| = \left|\frac{f(x)+g(x)-|f(x)-g(x)|}{2} - \frac{f(x_0)+g(x_0)-|f(x_0)-g(x_0)|}{2}\right|$$

$$= \left|\frac{(f(x)-f(x_0))+(g(x)-g(x_0))-(|f(x)-g(x)|-|f(x_0)-g(x_0)|)}{2}\right|$$

$$\leq \frac{|f(x)-f(x_0)|+|g(x)-g(x_0)|+||f(x)-g(x)|-|f(x_0)-g(x_0)||}{2}$$

$$\leq \frac{|f(x)-f(x_0)|+|g(x)-g(x_0)|+|(f(x)-g(x))-(f(x_0)-g(x_0))|}{2}$$

$$= \frac{|f(x)-f(x_0)|+|g(x)-g(x_0)|+|(f(x)-f(x_0))-(g(x)-g(x_0))|}{2}$$

$$\leq \frac{|f(x)-f(x_0)|+|g(x)-g(x_0)|+|f(x)-f(x_0)|+|g(x)-g(x_0)|}{2}$$

$$= |f(x)-f(x_0)|+|g(x)-g(x_0)| < \varepsilon.$$

Hence, $m(x)$ *conti.*

$$4.(3)\, M(x)=\max\{f(x),g(x)\}=\frac{f(x)+g(x)+|f(x)-g(x)|}{2}$$

for any given $x_0 \in I, \forall \varepsilon > 0,$

$\exists \delta_1 > 0, s.t. \forall x \in I : 0 < |x-x_0| < \delta_1,$

$|f(x)-f(x_0)| < \frac{\varepsilon}{2}$

$\exists \delta_2 > 0, s.t. \forall x \in I : 0 < |x-x_0| < \delta_2,$

$|g(x)-g(x_0)| < \frac{\varepsilon}{2}$

choose $\delta = \min\{\delta_1,\delta_2\} > 0,$ *then* $\forall x \in I : 0 < |x-x_0| < \delta,$

$$|M(x)-M(x_0)| = \left|\frac{f(x)+g(x)+|f(x)-g(x)|}{2} - \frac{f(x_0)+g(x_0)+|f(x_0)-g(x_0)|}{2}\right|$$

$$= \left|\frac{(f(x)-f(x_0))+(g(x)-g(x_0))+(|f(x)-g(x)|-|f(x_0)-g(x_0)|)}{2}\right|$$

$$\leq \frac{|f(x)-f(x_0)|+|g(x)-g(x_0)|+||f(x)-g(x)|-|f(x_0)-g(x_0)||}{2}$$

$$\leq \frac{|f(x)-f(x_0)|+|g(x)-g(x_0)|+|(f(x)-g(x))-(f(x_0)-g(x_0))|}{2}$$

$$= \frac{|f(x)-f(x_0)|+|g(x)-g(x_0)|+|(f(x)-f(x_0))-(g(x)-g(x_0))|}{2}$$

$$\leq \frac{|f(x)-f(x_0)|+|g(x)-g(x_0)|+|f(x)-f(x_0)|+|g(x)-g(x_0)|}{2}$$

$$= |f(x)-f(x_0)|+|g(x)-g(x_0)| < \varepsilon.$$

Hence, $M(x)$ *conti.*

$4.(4)\, u(x)=f(x)+g(x)+h(x)-\max\{f(x),g(x),h(x)\}-\min\{f(x),g(x),h(x)\}$

$denote \max\{f(x),g(x)\}\ by\ M_{fg}, \min\{f(x),g(x)\}\ by\ m_{fg}$

$by\ 4.(2)\ and\ 4.(3): M_{fg}, m_{fg}\ conti.$

$\Rightarrow u(x)=f(x)+g(x)+h(x)-\max\{M_{fg}(x),M_{fh}(x)\}-\min\{m_{fg}(x),m_{fh}(x)\}$

$=f(x)+g(x)+h(x)-M_{M_{fg}M_{fh}}(x)-m_{m_{fg}m_{fh}}(x)$

since $f(x),g(x),h(x),M_{M_{fg}M_{fh}}(x),m_{m_{fg}m_{fh}}(x)\ conti.$

for any given $x_0\in I,\forall\varepsilon>0,$

$\exists\delta_1>0, s.t.\forall x\in I:0<|x-x_0|<\delta_1,$

$|f(x)-f(x_0)|<\frac{\varepsilon}{5}$

$\exists\delta_2>0, s.t.\forall x\in I:0<|x-x_0|<\delta_2,$

$|g(x)-g(x_0)|<\frac{\varepsilon}{5}$

$\exists\delta_3>0, s.t.\forall x\in I:0<|x-x_0|<\delta_3,$

$|h(x)-h(x_0)|<\frac{\varepsilon}{5}$

$\exists\delta_4>0, s.t.\forall x\in I:0<|x-x_0|<\delta_4,$

$\left|M_{M_{fg}M_{fh}}(x)-M_{M_{fg}M_{fh}}(x_0)\right|<\frac{\varepsilon}{5}$

$\exists\delta_5>0, s.t.\forall x\in I:0<|x-x_0|<\delta_5,$

$\left|m_{m_{fg}m_{fh}}(x)-m_{m_{fg}m_{fh}}(x_0)\right|<\frac{\varepsilon}{5}$

$choose\ \delta=\min\{\delta_1,\delta_2,\delta_3,\delta_4,\delta_5\}>0, then\ \forall x\in I:0<|x-x_0|<\delta,$

$|u(x)-u(x_0)|=\left|\left(f(x)+g(x)+h(x)-M_{M_{fg}M_{fh}}(x)-m_{m_{fg}m_{fh}}(x)\right)-\left(f(x_0)+g(x_0)+h(x_0)-M_{M_{fg}M_{fh}}(x_0)-m_{m_{fg}m_{fh}}(x_0)\right)\right|$

$=|f(x)-f(x_0)|+|g(x)-g(x_0)|+|h(x)-h(x_0)|+\left|M_{M_{fg}M_{fh}}(x)-M_{M_{fg}M_{fh}}(x_0)\right|+\left|m_{m_{fg}m_{fh}}(x)-m_{m_{fg}m_{fh}}(x_0)\right|$

$<\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}=\varepsilon$

$Hence, u(x)\ conti.$

2.反证：假设$\exists c\in(m,M), s.t.\forall y\in I, f(y)\neq c$

$g(x)\triangleq f(x)-c$

因此$\forall y\in I, g(x)\neq 0$,又$f(x)$连续$\Rightarrow g(x)$连续

不妨设$\forall y\in I, g(x)>0.$

$\Rightarrow \inf\limits_{x\in I} g(x)\geq 0\Rightarrow \inf\limits_{x\in I} f(x)\geq c>m=\inf\limits_{x\in I} f(x)$,矛盾！

故$\forall c\in(m,M), s.t.\exists\xi\in I, f(\xi)=c.$

4.证：$g(x) \triangleq f(x)-x$

假设f没有不动点，即$\forall x\in[a,b], g(x)\neq 0$

因为$g(x)\in C[a,b]$,所以$g(x)$恒正或恒负

不妨设$\forall x\in[a,b], g(x)>0$

那么$f(b)>b$，但是$f(x)$值域为$[a,b]$, 矛盾！

故$\exists x\in[a,b], f(x)-x=g(x)=0$,即$f(x)$有不动点.

6.证：$g(x) \triangleq f(x)-\dfrac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$,

不妨设$f(x_1)\geq f(x_2)\geq\cdots\geq f(x_n)$

那么$g(x_1)\geq 0, g(x_n)\leq 0$

由零点存在性定理:

若$x_1>x_n, \exists\xi\in[x_n,x_1]\subseteq I$，使得$f(\xi)-\dfrac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}=g(\xi)=0$

即$f(\xi)=\dfrac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$

若$x_1<x_n, \exists\xi\in[x_1,x_n]\subseteq I$，使得$f(\xi)-\dfrac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}=g(\xi)=0$

即$f(\xi)=\dfrac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$

7.证：对于一个给定的x_1,

$\lim\limits_{|x|\to\infty} f(x)=+\infty \Rightarrow \exists M>0, s.t. \forall |x|>M$,

$f(x)>f(x_1)+1$,

对于$|x|\leq M$,由定理3.4.4 : $\exists x_0\in[-M,M], s.t. \forall x\in[-M,M]$

$f(x_0)\leq f(x)$

9.证：

假设$\forall x\in[a,b], f(x)\neq 0$

不妨设$\forall x\in[a,b], f(x)>0$

那么$\exists\lambda\in(0,1), s.t.$

$\forall x\in[a,b], \exists y\in[a,b], s.t.$

$f(y)\leq\lambda f(x)<f(x)$

$\Rightarrow \exists y\in[a,b], s.t. f(y)<\inf\limits_{x\in[a,b]} f(x)$,矛盾！

故f在区间$[a,b]$上有零点.