10.16 作业.

7. 证: "⇒": $\{x_{2n}\}$, $\{x_{2n-1}\}$ 为 $\{x_n\}$ 的子列 则 $\lim\limits_{n\to\infty} x_{2n} = \lim\limits_{n\to\infty} x_n = a$

$\lim\limits_{n\to\infty} x_{2n-1} = \lim\limits_{n\to\infty} x_n = a$

"⇐": 若 $n = 2k$. $\lim\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} x_{2k} = a$

若 $n = 2k-1$. $\lim\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} x_{2k-1} = a$

而 $\{2k : k\in\mathbb{N}\} \sqcup \{2k-1 : k\in\mathbb{N}\} = \mathbb{N}$,

则 $\lim\limits_{n\to\infty} x_n = a$

8. 证: $a > |b| \geq 0$

则 $\dfrac{\sqrt[n]{a^n+b^n}}{a} = \sqrt[n]{1+(\frac{b}{a})^n} = \left[1+(\frac{b}{a})^n\right]^{\frac{1}{n}}$

$\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{a^n+b^n}}{a} = \lim\limits_{n\to\infty} e^{\frac{1}{n}\ln[1+(\frac{b}{a})^n]}$ 其中 $|(\frac{b}{a})^n| = (\frac{|b|}{a})^n < 1 \Rightarrow (\frac{b}{a})^n \in (-1, 1)$

$\because \lim\limits_{n\to\infty} (\frac{b}{a})^n = 0$, $\therefore \exists N\in\mathbb{N}$. s.t. $\forall n > N$. 有 $(\frac{b}{a})^n \in (-\frac{1}{2}, \frac{1}{2})$

$\Rightarrow n > N$ 时. $\lim\limits_{n\to\infty} e^{\frac{1}{n}\ln(1-\frac{1}{2})} < \lim\limits_{n\to\infty} e^{\frac{1}{n}\ln[1+(\frac{b}{a})^n]} < \lim\limits_{n\to\infty} e^{\frac{1}{n}\ln(1+\frac{1}{2})}$

$\Rightarrow 1 \leq \lim\limits_{n\to\infty} e^{\frac{1}{n}\ln[1+(\frac{b}{a})^n]} \leq 1$

$\Rightarrow \lim\limits_{n\to\infty} \dfrac{\sqrt[n]{a^n+b^n}}{a} = 1 \Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{a^n+b^n} = a$

13. 证: $\lim\limits_{n\to\infty}(x_{n+1}-x_n) = a \Rightarrow \forall \varepsilon_0 > 0$,

$\exists N\in\mathbb{N}$. s.t. $\forall n \geq N$. 有 $(x_{n+1}-x_n) \in (a-\varepsilon_0, a+\varepsilon_0)$

则 $\forall n \geq N$.

$(n-N)(a-\varepsilon_0) < x_{n+1} - x_N < (n-N)(a+\varepsilon_0)$

$\Rightarrow \lim\limits_{n\to\infty}\left[\dfrac{n(a-\varepsilon_0)}{n} - \dfrac{N(a-\varepsilon_0)}{n}\right] < \lim\limits_{n\to\infty} \dfrac{x_{n+1}-x_N}{n} < \lim\limits_{n\to\infty}\left[\dfrac{n(a+\varepsilon_0)}{n} - \dfrac{N(a+\varepsilon_0)}{n}\right]$

$\Rightarrow a - \varepsilon_0 < \lim\limits_{n\to\infty} \dfrac{x_n}{n} < a + \varepsilon_0$

i.e. $\forall \varepsilon_0 > 0$. $\exists N\in\mathbb{N}$. s.t. $\forall n \geq N$. 有 $\left|\lim\limits_{n\to\infty} \dfrac{x_n}{n} - a\right| < \varepsilon_0$

$\Rightarrow \lim\limits_{n\to\infty} \dfrac{x_n}{n} = a$.

1.(4) $\lim\limits_{n\to\infty} \dfrac{3^n n^2 a + 2^n n^3 b}{3^n n^2 + 2^n n^3} = \lim\limits_{n\to\infty} \dfrac{a + (\frac{2}{3})^n n b}{1 + (\frac{2}{3})^n n} = \dfrac{\lim\limits_{n\to\infty} (\frac{2}{3})^n n b + a}{\lim\limits_{n\to\infty} (\frac{2}{3})^n n + 1} = a$

(8). $\lim\limits_{n\to\infty} \frac{1}{n^2} [1 + 3 + \dots + (2n-1)] = \lim\limits_{n\to\infty} \frac{1}{n^2} \frac{(1+2n-1)n}{2} = 1$

(10). $\lim\limits_{n\to\infty} (1-\frac{1}{2^2})(1-\frac{1}{3^2}) \cdots (1-\frac{1}{n^2}) = \lim\limits_{n\to\infty} \frac{2^2-1}{2^2} \cdot \frac{3^2-1}{3^2} \cdot \dots \cdot \frac{n^2-1}{n^2}$

$= \lim\limits_{n\to\infty} \frac{1\cdot 3}{2^2} \cdot \frac{2\cdot 4}{3^2} \cdot \frac{3\cdot 5}{4^2} \cdot \dots \cdot \frac{(n-1)(n+1)}{n^2}$

$= \lim\limits_{n\to\infty} \frac{1}{2} \cdot \frac{n+1}{n} = \frac{1}{2}$

(12). $\lim\limits_{n\to\infty} \left(\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \dots + \frac{1}{n(n+1)(n+2)} \right)$. 注意到: $\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right]$

$= \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{2}\left[\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right] = \frac{1}{2} \lim\limits_{n\to\infty} \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right] = \frac{1}{4}$.

5.(10) $\lim\limits_{n\to\infty} \sin(\pi\sqrt{n^2+1}) = \lim\limits_{n\to\infty} (-1)^n \sin(\pi\sqrt{n^2+1} - n\pi)$

$= \lim\limits_{n\to\infty} (-1)^n \sin(\frac{\pi}{\sqrt{n^2+1}+n})$

$\lim\limits_{n\to\infty} \left| (-1)^n \sin(\frac{\pi}{\sqrt{n^2+1}+n}) \right| = \lim\limits_{n\to\infty} \left| \sin(\frac{\pi}{\sqrt{n^2+1}+n}) \right| \le \lim\limits_{n\to\infty} \left| \frac{\pi}{\sqrt{n^2+1}+n} \right| = 0$

$\Rightarrow \lim\limits_{n\to\infty} \sin(\pi\sqrt{n^2+1}) = 0$

(11). $\lim\limits_{n\to\infty} (-1)^n \sin(\pi\sqrt{n^2+n}) = \lim\limits_{n\to\infty} \sin(\pi\sqrt{n^2+n} - \pi n)$

$= \lim\limits_{n\to\infty} \sin(\pi \frac{n}{\sqrt{n^2+n}+n})$

$= \lim\limits_{n\to\infty} \sin(\pi \frac{1}{\sqrt{1+\frac{1}{n}}+1})$ ~~其 sin x~~

$= \sin\frac{\pi}{2} = 1$

1.(3). $0 < x_1 < \frac{1}{a}$. $x_{n+1} = x_n(2 - a x_n)$. 记 $y_n = a x_n$. $\Rightarrow 0 < y_1 < 1$

$y_{n+1} = y_n(2 - y_n) \Rightarrow |y_{n+1} - 1| = |y_n - 1|^2 = \dots = |y_1 - 1|^{2^n}$

$|y_1 - 1| \in (0,1) \Rightarrow \lim\limits_{n\to\infty} |y_{n+1} - 1| = \lim\limits_{n\to\infty} |y_1 - 1|^{2^n} = 0 \Rightarrow \lim\limits_{n\to\infty} y_n = 1. \Rightarrow \lim\limits_{n\to\infty} x_n = \frac{1}{a}$

(7). $x_1 = 1$. $x_2 = \sqrt{1+\sqrt{1}}$, $x_3 = \sqrt{1+\sqrt{1+\sqrt{1}}}$, $\dots$, $x_n = \underbrace{\sqrt{1+\sqrt{1+\dots+\sqrt{1}}}}_{n个根号} =$ ~~x_n~~

$\Rightarrow x_{n+1} =$ ~~$\sqrt{1+\dots x_n}$~~ $\sqrt{1+x_n}$ $x_n > 1$. $\Rightarrow x_{n+1} > \sqrt{2} \dots$

$|x_{n+1} - x_n| = \left|\sqrt{1+x_n} - \sqrt{1+x_{n-1}}\right| = \left|\frac{x_n - x_{n-1}}{\sqrt{1+x_n} + \sqrt{1+x_{n-1}}}\right| < \frac{1}{2}|x_n - x_{n-1}| < \dots < \frac{1}{2^{n-1}}|x_2 - x_1|$

~~$\Rightarrow \lim\limits_{n\to\infty} |x_{n+1} - x_n| < \lim\limits_{n\to\infty} \frac{1}{2^{n-1}} |x_2 - x_1| = \lim\limits_{n\to\infty} \frac{\sqrt{2}-1}{2^{n-1}} = 0$~~

~~$\Rightarrow \forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall n > N$ 有 $|x_{n+1} - x_n|$ $\forall m > n > N$ 有~~

~~$\forall n > N$ 有 $|x_{n+1} - x_n| <$~~

$\Rightarrow \forall m > n$, $|x_m - x_n| = |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \dots + (x_{n+1} - x_n)|$

$$\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n|$$

$$= \left(\frac{1}{2^{m-2}} + \frac{1}{2^{m-3}} + \dots + \frac{1}{2^{n-1}}\right)(\sqrt{2}-1)$$

$$= \frac{\frac{1}{2^{n-1}} - \frac{1}{2^{m-1}}}{1-\frac{1}{2}} (\sqrt{2}-1) < \frac{1}{2^{n-2}}(\sqrt{2}-1)$$

$\Rightarrow \lim\limits_{\substack{n\to\infty \\ m>n}} |x_m - x_n| \le \lim\limits_{n\to\infty} \frac{1}{2^{n-1}}(\sqrt{2}-1) = 0$

由 Cauchy convergence criterion: $\{x_n\}$ converge.

记 $\lim\limits_{n\to\infty} x_n = a > 1$. $\Rightarrow \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} \sqrt{1+x_n} \Rightarrow a = \sqrt{1+a} \Rightarrow a^2 - a - 1 = 0 \Rightarrow a = \frac{1+\sqrt{5}}{2}$

$\Rightarrow \lim\limits_{n\to\infty} x_n = \frac{1+\sqrt{5}}{2}$

5.(3). 证: $\limsup\limits_{n\to\infty} y_{n+1} = \limsup\limits_{n\to\infty} \sqrt{y_n + x_n} = \sqrt{\limsup\limits_{n\to\infty} y_n + \limsup\limits_{n\to\infty} x_n}$

类似地, $\liminf\limits_{n\to\infty} y_{n+1} = \sqrt{\liminf\limits_{n\to\infty} y_n + \liminf\limits_{n\to\infty} x_n}$

则 $\lim\limits_{n\to\infty} y_n$ 存在 $\Leftrightarrow \limsup\limits_{n\to\infty} y_{n+1} = \liminf\limits_{n\to\infty} y_{n+1} \Leftrightarrow \sqrt{\limsup\limits_{n\to\infty} y_n + \limsup\limits_{n\to\infty} x_n} = \sqrt{\liminf\limits_{n\to\infty} y_n + \liminf\limits_{n\to\infty} x_n}$

$\Leftrightarrow \limsup\limits_{n\to\infty} x_n = \liminf\limits_{n\to\infty} x_n \Leftrightarrow \lim\limits_{n\to\infty} x_n$ 存在

8.(1). 证: $\because \{x_n\}$ 有界, 记 $\sup\{x_n\} = M$, $\inf\{x_n\} = m$.

$\forall \varepsilon > 0$ $[m, M] = \left(\bigcup_{k=1}^{[\frac{M-m}{\varepsilon}]+1} [m+(k-1)\varepsilon,\ m+k\varepsilon]\right) \cup \{M\}$

由抽屉原理: 在这 $[\frac{M-m}{\varepsilon}]+2$ 个集合中必存在一个集合有 $\{x_n\}$ 中无穷多个数. 不妨记为 I.

$\forall x_n, x_m \in I$. $|x_m - x_n| \le \varepsilon$ 由柯西收敛准则: $\{x_n\}$ 收敛.

记 $\lim\limits_{n\to\infty} x_n = a \Rightarrow 1 = \lim\limits_{n\to\infty} (x_{n+1} + 2x_n) = a + 2a = 3a \Rightarrow a = \frac{1}{3}$ i.e. $\lim\limits_{n\to\infty} x_n = \frac{1}{3}$

8.(4). ① $\because x_n = O(\frac{1}{n}) \Rightarrow \lim\limits_{n\to\infty} \frac{x_n}{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{O(\frac{1}{n})}{\frac{1}{n}} = C$ 为常数. 由 $O(x)$ 定义.

$\Rightarrow \lim\limits_{n\to\infty} n x_n = C$ 常数 $\Rightarrow \lim\limits_{n\to\infty} n x_n$ 存在 $\Rightarrow \lim\limits_{n\to\infty} x_n = \frac{C}{n}$.

② $\Rightarrow \lim\limits_{n\to\infty} n(x_{2n} + x_n) = n\left(\frac{C}{2n} + \frac{C}{n}\right) = c$. $\Rightarrow C = \frac{2}{3}c$. $\Rightarrow \lim\limits_{n\to\infty} x_n = \frac{2c}{3n}$.

$$1.(1)\ \forall\varepsilon>0, choose\ \delta=\sqrt{4+\varepsilon}-2>0, s.t. \forall x, 0<\left|x-2\right|<\delta,$$

$$\left|x^2-4\right|=|x-2||x+2|<\delta|x+2|=\varepsilon$$

$$\Rightarrow \lim_{x\to 2}x^2=4.$$

$$1.(2)\ \forall\varepsilon>0, choose\ \delta=\frac{-1+\sqrt{1+24\varepsilon}}{2}>0, s.t. \forall x, 0<\left|x-1\right|<\delta,$$

$$\left|\frac{x}{2x^2+1}-\frac{1}{3}\right|=\left|\frac{(2x-1)(x-1)}{3(2x^2+1)}\right|\le\frac{|2x-1||x-1|}{3}<\frac{\delta(1+2\delta)}{3}=\varepsilon$$

$$\Rightarrow \lim_{x\to 1}\frac{x}{2x^2+1}=\frac{1}{3}.$$

$$1.(5)\ \forall\varepsilon>0, choose\ \delta=\frac{\varepsilon}{2}>0, s.t. \forall x, 0<\left|x-\frac{\pi}{2}\right|<\delta,$$

$$\left|(2x-\pi)\cos\frac{x-\pi}{2x-\pi}\right|\le|2x-\pi|=2\left|x-\frac{\pi}{2}\right|<2\delta=\varepsilon$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}}(2x-\pi)\cos\frac{x-\pi}{2x-\pi}=0.$$

2.(1)

$\bullet$*if* $x_0=0$, *then we need to show*: $\lim_{x\to 0}x^n=0.$

$$\forall\varepsilon>0, choose\ \delta=\min\left\{\frac{1}{2},\varepsilon\right\}\in\left(0,\frac{1}{2}\right], s.t. \forall x, 0<|x|<\delta,$$

$$\left|x^n\right|=|x|<\delta\le\varepsilon.$$

$$\Rightarrow \lim_{x\to 0}x^n=0.$$

$\bullet$*if* $x_0\ne 0$, without loss of generality:we let $x_0=1$, otherwise we replace x by $\dfrac{x}{x_0}$

then we need to show: $\lim_{x\to 1}x^n=1.$

$$\forall\varepsilon>0, choose\ \delta=\min\left\{\frac{1}{2},\frac{-1+\sqrt{1+\frac{4\varepsilon}{n}}}{2}\right\}\in\left(0,\frac{1}{2}\right], s.t. \forall x, 0<\left|x-1\right|<\delta,$$

$$\left|x^n-1\right|=|x-1|\left|1+x+x^2+\cdots+x^{n-1}\right|<\delta\left|1+x+x^2+\cdots+x^{n-1}\right|$$

$$\le\delta\left(1+|x|+\left|x^2\right|+\cdots+\left|x^{n-1}\right|\right)\le\delta\left[1+(n-1)|x|\right]<\delta\left[1+(n-1)(1+\delta)\right]$$

$$=(n-1)\delta^2+n\delta<n\left(\delta^2+\delta\right)\le\varepsilon.$$

$$\Rightarrow \lim_{x\to 1}x^n=1.$$

Hence, $\lim_{x\to x_0}x^n=x_0^n.$

3.(2)

$\bullet if\ \ a=0, then\ we\ need\ to\ show: \lim_{x\to x_0}\sqrt[3]{f(x)}=0$

$we\ know\ that: \lim_{x\to x_0} f(x)=0$

$thus \forall \varepsilon>0$,

$\exists \delta_1>0, s.t. \forall x, 0<|x-x_0|<\delta_1$,

$|f(x)|<\varepsilon^3$

$\Rightarrow \left|\sqrt[3]{f(x)}\right|<\varepsilon$

$\Rightarrow \lim_{x\to x_0}\sqrt[3]{f(x)}=0$

$\bullet if\ \ a\neq 0$, without loss of generality:we let $a=1$, otherwise we replace $f(x)$ by $\dfrac{f(x)}{a}$

$then\ we\ need\ to\ show: \lim_{x\to x_0}\sqrt[3]{f(x)}=1$

$we\ know\ that: \lim_{x\to x_0} f(x)=1$

$thus \forall \varepsilon>0$,

$\exists \delta_2>0, s.t. \forall x, 0<|x-x_0|<\delta_2$,

$|f(x)-1|<\dfrac{1}{2}\Rightarrow f(x)\in\left(\dfrac{1}{2},\dfrac{3}{2}\right)$

$\exists \delta_3>0, s.t. \forall x, 0<|x-x_0|<\delta_3$,

$|f(x)-1|<\varepsilon$

$\Rightarrow choose\ \delta=\min\{\delta_2,\delta_3\}>0, s.t. \forall x, 0<|x-x_0|<\delta$,

$$\Rightarrow \left|\sqrt[3]{f(x)}-1\right|=\frac{|f(x)-1|}{\left|\left(\sqrt[3]{f(x)}\right)^2+\sqrt[3]{f(x)}+1\right|}=\frac{|f(x)-1|}{\left(\sqrt[3]{f(x)}\right)^2+\sqrt[3]{f(x)}+1}<|f(x)-1|<\varepsilon$$

$\Rightarrow \lim_{x\to x_0}\sqrt[3]{f(x)}=1$

$Hence, \lim_{x\to x_0}\sqrt[3]{f(x)}=\sqrt[3]{a}.$

$$6.(1)\lim_{x\to 0}\frac{(x-1)^3-2x-1}{x^3+x-2}=\lim_{x\to 0}\frac{x^3-3x^2+x-2}{x^3+x-2}=\lim_{x\to 0}\frac{1-\frac{3}{x}+\frac{1}{x^2}-\frac{2}{x^3}}{1+\frac{1}{x^2}-\frac{2}{x^3}}=\frac{\lim\limits_{x\to 0}\left(1-\frac{3}{x}+\frac{1}{x^2}-\frac{2}{x^3}\right)}{\lim\limits_{x\to 0}\left(1+\frac{1}{x^2}-\frac{2}{x^3}\right)}=1$$

$$6.(3)\lim_{x\to 0}\frac{(1+x)(1+2x)(1-3x)-1}{2x^3+x^2}=\lim_{x\to 0}\frac{\left(\frac{1}{x}+1\right)\left(\frac{1}{x}+2\right)\left(\frac{1}{x}-3\right)-\frac{1}{x^3}}{2+\frac{1}{x}}=\frac{\lim\limits_{x\to 0}\left[\left(\frac{1}{x}+1\right)\left(\frac{1}{x}+2\right)\left(\frac{1}{x}-3\right)-\frac{1}{x^3}\right]}{\lim\limits_{x\to 0}\left(2+\frac{1}{x}\right)}$$

$$=\frac{\lim\limits_{x\to 0}\left(\frac{1}{x}+1\right)\lim\limits_{x\to 0}\left(\frac{1}{x}+2\right)\lim\limits_{x\to 0}\left(\frac{1}{x}-3\right)-\lim\limits_{x\to 0}\frac{1}{x^3}}{\lim\limits_{x\to 0}\left(2+\frac{1}{x}\right)}=\frac{1\cdot 2\cdot(-3)}{2}=-3.$$

$$6.(5)\lim_{x\to 1}\frac{x^3-3x+2}{x^4-4x+3}=\lim_{x\to 1}\frac{(x-1)^2(x+2)}{(x-1)^2(x^2+2x+3)}=\lim_{x\to 1}\frac{x+2}{x^2+2x+3}=\frac{\lim\limits_{x\to 1}(x+2)}{\lim\limits_{x\to 1}(x^2+2x+3)}=\frac{1}{2}$$

$$6.(7)\lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}=\lim_{x\to 4}\frac{2(x-4)(\sqrt{x}+2)}{(\sqrt{1+2x}+3)(x-4)}=\frac{\lim\limits_{x\to 4}2(\sqrt{x}+2)}{\lim\limits_{x\to 4}(\sqrt{1+2x}+3)}=\frac{4}{3}$$

$$6.(9)\lim_{x\to 1}\frac{\sqrt{2-x}-\sqrt{x}}{\sqrt[3]{2-x}-\sqrt[3]{x}}=\lim_{x\to 1}\frac{2(1-x)}{\sqrt{2-x}+\sqrt{x}}\frac{\left(\sqrt[3]{2-x}\right)^2+\sqrt[3]{2-x}\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2}{2(1-x)}=\lim_{x\to 1}\frac{\left(\sqrt[3]{2-x}\right)^2+\sqrt[3]{2-x}\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2}{\sqrt{2-x}+\sqrt{x}}$$

$$=\frac{\lim\limits_{x\to 1}\left[\left(\sqrt[3]{2-x}\right)^2+\sqrt[3]{2-x}\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2\right]}{\lim\limits_{x\to 1}\left(\sqrt{2-x}+\sqrt{x}\right)}=\frac{\lim\limits_{x\to 1}\left(\sqrt[3]{2-x}\right)^2+\lim\limits_{x\to 1}\sqrt[3]{2-x}\sqrt[3]{x}+\lim\limits_{x\to 1}\left(\sqrt[3]{x}\right)^2}{\lim\limits_{x\to 1}\sqrt{2-x}+\lim\limits_{x\to 1}\sqrt{x}}=\frac{3}{2}$$

$$10.(2)\lim_{n\to\infty}\frac{\sqrt[m]{1+\sin\frac{\pi}{n}}-1}{\sin\frac{\pi}{n}}\overset{\text{by heine Thm}}{\underset{x\triangleq\frac{\pi}{n}\to 0}{=}}\lim_{x\to 0}\frac{\sqrt[m]{1+\sin x}-1}{\sin x}=\lim_{x\to 0}\frac{\sqrt[m]{1+\sin x}-1}{(1+\sin x)-1}$$

$$=\lim_{x\to 0}\frac{\sqrt[m]{1+\sin x}-1}{\left(\sqrt[m]{1+\sin x}-1\right)\left[1+\left(\sqrt[m]{1+\sin x}\right)+\left(\sqrt[m]{1+\sin x}\right)^2+\cdots+\left(\sqrt[m]{1+\sin x}\right)^{m-1}\right]}$$

$$=\lim_{x\to 0}\frac{1}{1+\left(\sqrt[m]{1+\sin x}\right)+\left(\sqrt[m]{1+\sin x}\right)^2+\cdots+\left(\sqrt[m]{1+\sin x}\right)^{m-1}}$$

$$=\frac{1}{1+\left(\lim\limits_{x\to 0}\sqrt[m]{1+\sin x}\right)+\left(\lim\limits_{x\to 0}\sqrt[m]{1+\sin x}\right)^2+\cdots+\left(\lim\limits_{x\to 0}\sqrt[m]{1+\sin x}\right)^{m-1}}$$

$$=\frac{1}{\underbrace{1+1+1+\cdots+1}_{m\text{个}}}=\frac{1}{m}$$

$$10.(4)\lim_{n\to\infty}\ln\left(1+\sqrt{n^2+1}\right)-\ln n\overset{\text{by heine Thm}}{=}\lim_{x\to\infty}\ln\left(1+\sqrt{x^2+1}\right)-\ln x=\lim_{x\to\infty}\ln\frac{1+\sqrt{x^2+1}}{x}$$

$$=\ln\left(\lim_{x\to\infty}\frac{1+\sqrt{x^2+1}}{x}\right)=\ln\left(\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\sqrt{1+\frac{1}{x^2}}\right)=\ln\left(\lim_{x\to\infty}\frac{1}{x}+\sqrt{1+\lim_{x\to\infty}\frac{1}{x^2}}\right)=0.$$