5.(1)由伯努利不等式: $1>\sqrt{1-\frac{1}{n}}=\left(1-\frac{1}{n}\right)^{\frac{1}{2}}>1-\frac{1}{2n}.$

$\lim\limits_{n\to\infty}\left(1-\frac{1}{2n}\right)=1$,由两边夹法则,可知: $\lim\limits_{n\to\infty}\sqrt{1-\frac{1}{n}}=1.$

5.(2) $0\le\left|\frac{\sin n!}{\sqrt{n}}\right|\le\left|\frac{1}{\sqrt{n}}\right|$

$\lim\limits_{n\to\infty}\left|\frac{1}{\sqrt{n}}\right|=0$,由两边夹法则,可知: $\lim\limits_{n\to\infty}\frac{\sin n!}{\sqrt{n}}=0.$

5.(4) $\sqrt[n]{n}<\sqrt[n]{n\log_2 n}<\sqrt[n]{n\log_2 2^n}=\sqrt[\frac{n}{2}]{n}$

断言: $\lim\limits_{n\to\infty}\sqrt[n]{n}=1,\lim\limits_{n\to\infty}\sqrt[\frac{n}{2}]{n}=1.$

$\forall\varepsilon>0,\exists N=\left[\frac{2}{\varepsilon^2}\right]+2,s.t.\forall n>N,$

$$(1+\varepsilon)^n=1+C_n^1\varepsilon+C_n^2\varepsilon^2+\cdots+C_n^n\varepsilon^n>C_n^2\varepsilon^2=\frac{n(n-1)}{2}\varepsilon^2>\frac{n\left(\left[\frac{2}{\varepsilon^2}\right]+1\right)}{2}\varepsilon^2>n$$

$\Rightarrow\left|\sqrt[n]{n}-1\right|<\varepsilon\Rightarrow\lim\limits_{n\to\infty}\sqrt[n]{n}=1.$

$\forall\varepsilon>0,\exists N=\left[\frac{6}{\varepsilon^3}\right]+4,s.t.\forall n>N,$

$$(1+\varepsilon)^n=1+C_n^1\varepsilon+C_n^2\varepsilon^2+\cdots+C_n^n\varepsilon^n>C_n^3\varepsilon^3=\frac{n(n-1)(n-2)}{6}\varepsilon^3>\frac{n^2(n-3)}{6}\varepsilon^3>n^2$$

$\Rightarrow\left|\sqrt[\frac{n}{2}]{n}-1\right|=\sqrt[\frac{n}{2}]{n}-1<\varepsilon\Rightarrow\lim\limits_{n\to\infty}\sqrt[\frac{n}{2}]{n}=1.$

$\Rightarrow$由两边夹法则,可知: $\lim\limits_{n\to\infty}\sqrt[n]{n\log_2 n}=0.$

5.(5) $1=\frac{n}{\sqrt{n^2}}>\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}>\frac{n}{\sqrt{n^2+n}}=\frac{1}{\sqrt{1+\frac{1}{n}}}$

$\lim\limits_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}=\frac{1}{\lim\limits_{n\to\infty}\sqrt{1+\frac{1}{n}}}=1$,由两边夹法则,可知: $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}=1.$

5.(6) $2+\frac{1}{n}=\frac{2n+1}{n}>\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{(n+1)^2}}>\frac{2n+1}{\sqrt{(n+1)^2}}=\frac{2n+1}{n+1}=2-\frac{1}{n+1}$

$\lim\limits_{n\to\infty}2+\frac{1}{n}=2,\lim\limits_{n\to\infty}2-\frac{1}{n+1}=2$,由两边夹法则,可知: $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{(n+1)^2}}=2.$

5.(7) $\frac{1}{\sqrt{\frac{n}{2}-1}}=\frac{1}{\sqrt[n]{\left(\frac{n}{2}-1\right)^{\frac{n}{2}}}}\ge\frac{1}{\sqrt[n]{\left[\frac{n}{2}\right]^{n-\left[\frac{n}{2}\right]}}}>\frac{1}{\sqrt[n]{\left[\frac{n}{2}\right]!\left[\frac{n}{2}\right]^{n-\left[\frac{n}{2}\right]}}}>\frac{1}{\sqrt[n]{n!}}>0.$

$\lim\limits_{n\to\infty}\frac{1}{\sqrt{\frac{n}{2}-1}}=0$,由两边夹法则,可知: $\lim\limits_{n\to\infty}\frac{1}{\sqrt[n]{n!}}=0.$

$$7.\max_{1\le k\le m}\{a_k\}=\sqrt[n]{\left[\max_{1\le k\le m}\{a_k\}\right]^n}\le\sqrt[n]{a_1^n+a_2^n+\cdots+a_m^n}$$

$$\le\sqrt[n]{\left[\max_{1\le k\le m}\{a_k\}\right]^n+\left[\max_{1\le k\le m}\{a_k\}\right]^n+\cdots+\left[\max_{1\le k\le m}\{a_k\}\right]^n}$$

$$=\sqrt[n]{m\left[\max_{1\le k\le m}\{a_k\}\right]^n}=\max_{1\le k\le m}\{a_k\}\sqrt[n]{m}$$

$\lim\limits_{n\to\infty}\max\limits_{1\le k\le m}\{a_k\}\sqrt[n]{m}=\max\limits_{1\le k\le m}\{a_k\}$,由两边夹法则, 可知：

$$\lim_{n\to\infty}\sqrt[n]{a_1^n+a_2^n+\cdots+a_m^n}=\max_{1\le k\le m}\{a_k\}.$$

8.证明：

$$\left.\begin{array}{l}x_n>0,n=1,2,\cdots\\ \lim\limits_{n\to\infty}x_n=a>0\end{array}\right\}\Rightarrow\max_{k\ge1}\{x_k\},\min_{k\ge1}\{x_k\}>0$$

$$\sqrt[n]{\max_{k\ge1}\{x_k\}}\ge\sqrt[n]{x_n}\ge\sqrt[n]{\min_{k\ge1}\{x_k\}}$$

$\lim\limits_{n\to\infty}\sqrt[n]{\max\limits_{k\ge1}\{x_k\}}=1,\lim\limits_{n\to\infty}\sqrt[n]{\min\limits_{k\ge1}\{x_k\}}=1$,由两边夹法则, 可知:$\lim\limits_{n\to\infty}\sqrt[n]{x_n}=1.$

9.(2)证明：不妨设$a=1$,否则用$\dfrac{x_n}{a}$代替x_n.

只需证：$\forall b>0,\lim\limits_{n\to\infty}\log_b x_n=0$

只需证:$\lim\limits_{n\to\infty}\ln x_n=0.$

因为$\lim\limits_{n\to\infty}x_n=1\Rightarrow\forall\varepsilon>0,\dfrac{\varepsilon}{1+\varepsilon}>0,\exists N\in\mathbb{N},\forall n>N,|x_n-1|<\varepsilon_0=\dfrac{\varepsilon}{1+\varepsilon}<\varepsilon,\dfrac{\varepsilon_0}{1-\varepsilon_0}=\varepsilon$

$$|\ln x_n|<\max\left\{\ln\frac{1}{1-\varepsilon_0},\ln(1+\varepsilon_0)\right\}\le\max\left\{\frac{1}{1-\varepsilon_0}-1,\varepsilon_0\right\}=\max\left\{\frac{\varepsilon_0}{1-\varepsilon_0},\varepsilon_0\right\}\le\varepsilon$$

$$\Rightarrow\lim_{n\to\infty}\ln x_n=0.$$

(3) *proof* :

WLOG : $a=1$,

need to show : $\forall b>0,\lim\limits_{n\to\infty}x_n^b=1.$

$$\lim_{n\to\infty}x_n=1\Rightarrow\lim_{n\to\infty}x_n^b=\lim_{n\to\infty}e^{b\ln x_n}=e^{b\lim\limits_{n\to\infty}\ln x_n}\overset{by\ 9.(2)}{=}e^0=1.$$

9.(5)

proof:

$-1\le x_n\le 1\Rightarrow -1\le a\le 1,$

$\bullet when\ a\ne \pm 1,$

$\forall \varepsilon>0,\exists N\in\mathbb{N},s.t.\forall n>N,\left|x_n-a\right|<\varepsilon.$

$\bullet Lemma: when -\frac{\pi}{2}\le z_n\le\frac{\pi}{2},\lim_{n\to\infty}z_n=0\Leftrightarrow\lim_{n\to\infty}\sin z_n=0.$

proof:

$"\Rightarrow":\forall\varepsilon>0,\exists N\in\mathbb{N},s.t.\forall n>N,\left|z_n\right|<\varepsilon\Rightarrow\left|\sin z_n\right|<\left|z_n\right|<\varepsilon.$

$"\Leftarrow":\forall\varepsilon>0,\exists N\in\mathbb{N},s.t.\forall n>N,\left|\sin z_n\right|<\frac{2}{\pi}\varepsilon\Rightarrow\left|z_n\right|\le\frac{\pi}{2}\left|\sin z_n\right|<\varepsilon.$

$Hence,\lim_{n\to\infty}z_n=0\Leftrightarrow\lim_{n\to\infty}\sin z_n=0.$

$\bullet y_n\triangleq\frac{1}{x_n^2}>1,b\triangleq\frac{1}{a^2}>1.$

$then\lim_{n\to\infty}\arcsin x_n-\arcsin a=0\Leftrightarrow\lim_{n\to\infty}\sin\left(\arcsin x_n-\arcsin a\right)=0$

$$\Leftrightarrow\lim_{n\to\infty}\frac{x_n}{\sqrt{1-x_n^2}}=\frac{a}{\sqrt{1-a^2}}\Leftrightarrow\lim_{n\to\infty}\frac{1}{\sqrt{\frac{1}{x_n^2}-1}}=\frac{1}{\sqrt{\frac{1}{a^2}-1}}\Leftrightarrow\lim_{n\to\infty}\frac{1}{\sqrt{y_n-1}}=\frac{1}{\sqrt{b-1}}$$

$\bullet Claim:\lim_{n\to\infty}y_n=b.$

$\forall\varepsilon>0,$

$\exists N_1\in\mathbb{N},s.t.\forall n>N_1,\left|x_n-a\right|<\frac{a}{2}\Rightarrow\left|\frac{a}{2}\right|<\left|x_n\right|<\left|\frac{3a}{2}\right|$

$\exists N_2\in\mathbb{N},s.t.\forall n>N_2,\left|x_n-a\right|<\frac{\left|a\right|^3}{10}\varepsilon$

$\exists N=\max\left\{N_1,N_2\right\},\forall n>N,$

$$\left|\frac{1}{x_n^2}-\frac{1}{a^2}\right|=\left|\frac{a^2-x_n^2}{x_n^2a^2}\right|=\left|\frac{\left(a-x_n\right)\left(a+x_n\right)}{x_n^2a^2}\right|\le\frac{\left|a-x_n\right|\left|a+x_n\right|}{x_n^2a^2}\le\frac{\left|a-x_n\right|\left(\left|a\right|+\left|x_n\right|\right)}{x_n^2a^2}$$

$$<\frac{\left|a-x_n\right|\left(\left|a\right|+\left|\frac{3a}{2}\right|\right)}{\left(\frac{a}{2}\right)^2a^2}=\frac{10}{\left|a\right|^3}\cdot\left|a-x_n\right|<\varepsilon$$

$\Rightarrow\lim_{n\to\infty}y_n=b.$

$\bullet need\ to\ show:\lim_{n\to\infty}y_n=b\Rightarrow\lim_{n\to\infty}\frac{1}{\sqrt{y_n-1}}=\frac{1}{\sqrt{b-1}}$

$\forall\varepsilon>0,$

$\exists N_1\in\mathbb{N},s.t.\forall n>N_1,\left|b-y_n\right|<\frac{1}{2}b-\frac{1}{2}\Rightarrow y_n>\frac{1}{2}b+\frac{1}{2}$

$\exists N_2\in\mathbb{N},s.t.\forall n>N_2,\left|b-y_n\right|<\frac{1}{2}\left(\sqrt{2b-2}+\sqrt{b-1}\right)\cdot\left(b-1\right)\cdot\varepsilon$

$\exists N=\max\left\{N_1,N_2\right\},\forall n>N,$

$$\left|\frac{1}{\sqrt{y_n-1}}-\frac{1}{\sqrt{b-1}}\right|=\left|\frac{\sqrt{b-1}-\sqrt{y_n-1}}{\sqrt{\left(y_n-1\right)\left(b-1\right)}}\right|=\left|\frac{\left(\sqrt{b-1}-\sqrt{y_n-1}\right)\left(\sqrt{b-1}+\sqrt{y_n-1}\right)}{\left(\sqrt{b-1}+\sqrt{y_n-1}\right)\cdot\sqrt{\left(y_n-1\right)\left(b-1\right)}}\right|$$

$$=\left|\frac{b-y_n}{\left(\sqrt{b-1}+\sqrt{y_n-1}\right)\cdot\sqrt{\left(y_n-1\right)\left(b-1\right)}}\right|<\left|\frac{b-y_n}{\left(\sqrt{b-1}+\sqrt{y_n-1}\right)\cdot\sqrt{\left(y_n-1\right)\left(b-1\right)}}\right|$$

$$<\frac{\left|b-y_n\right|}{\left(\sqrt{b-1}+\sqrt{\frac{1}{2}b+\frac{1}{2}-1}\right)\cdot\sqrt{\left(\frac{1}{2}b+\frac{1}{2}-1\right)\left(b-1\right)}}$$

$$=\frac{2\left|b-y_n\right|}{\left(\sqrt{2b-2}+\sqrt{b-1}\right)\cdot\left(b-1\right)}<\varepsilon.$$

$$\Rightarrow\lim_{n\to\infty}\frac{1}{\sqrt{y_n-1}}=\frac{1}{\sqrt{b-1}}.$$

$\bullet$ *when* $a=\pm 1, WLOG: let\ a=1.$

$$\lim_{n\to\infty}\arcsin x_n-\arcsin a=0\Leftrightarrow\lim_{n\to\infty}\arcsin x_n-\frac{\pi}{2}=0$$

$$\Leftrightarrow\lim_{n\to\infty}\sin\left(\arcsin x_n-\frac{\pi}{2}\right)=0\Leftrightarrow\lim_{n\to\infty}\cos\arcsin x_n=0$$

$$\Leftrightarrow\lim_{n\to\infty}\sqrt{1-x_n^2}=0$$

$\forall\varepsilon>0,$

$$\exists N\in\mathbb{N}, s.t.\forall n>N,\left|1-x_n\right|<\frac{\varepsilon^2}{2}$$

$$\Rightarrow\left|\sqrt{1-x_n^2}\right|=\left|\sqrt{(1-x_n)(1+x_n)}\right|\le\left|\sqrt{2(1-x_n)}\right|<\left|\sqrt{2\cdot\frac{\varepsilon^2}{2}}\right|=\varepsilon$$

$$\Rightarrow\lim_{n\to\infty}\arcsin x_n=\arcsin a$$

$\bullet$ *Hence*, $\lim_{n\to\infty}\arcsin x_n=\arcsin a.$

$$11.(1)\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+m)}=\lim_{n\to\infty}\frac{1}{m}\sum_{k=1}^{n}\frac{1}{k}-\frac{1}{k+m}=\frac{1}{m}\left(\frac{1}{1}+\frac{1}{2}+\cdots\frac{1}{m}\right).$$

$$11.(3)\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^3+6k^2+11k+5}{(k+3)!}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(k+3)(k+2)(k+1)}{(k+3)!}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k!}=\sum_{k=1}^{\infty}\frac{1}{k!}$$

$$\text{since } e^x=1+\sum_{k=1}^{\infty}\frac{x^k}{k!}\Rightarrow\sum_{k=1}^{\infty}\frac{1}{k!}=e-1\Rightarrow\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^3+6k^2+11k+5}{(k+3)!}=e-1.$$

Exercise 2.3

1.*proof* :

$$\forall a>1,\ln a>0,\forall M>0,\exists N=\left[\left(\frac{2M}{\ln a}\right)^2\right]+1,s.t.\forall n>N,$$

$$\frac{n}{\log_a n}=\frac{n\ln a}{\ln n}=\frac{n\ln a}{2\ln\sqrt{n}}>\frac{n\ln a}{2\left(\sqrt{n}-1\right)}>\frac{\sqrt{n}\ln a}{2}>M.$$

$$\Rightarrow\lim_{n\to\infty}\frac{n}{\log_a n}=+\infty.$$

2.(2)

$$\forall M>1,\exists N=\left[e^{(a+b)M}\right],s.t.\forall n>N,$$

$$\frac{1}{a+b}+\frac{1}{2a+b}+\frac{1}{3a+b}+\cdots+\frac{1}{na+b}>\frac{1}{a+b}+\frac{1}{2a+2b}+\frac{1}{3a+3b}+\cdots+\frac{1}{na+nb}$$

$$=\frac{1}{a+b}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)>\frac{1}{a+b}\left[\ln(1+1)+\ln\left(\frac{1}{2}+1\right)+\ln\left(\frac{1}{3}+1\right)+\cdots+\ln\left(\frac{1}{n}+1\right)\right]$$

$$=\frac{1}{a+b}\left[\ln 2+\ln\frac{3}{2}+\ln\frac{4}{3}+\cdots+\ln\frac{n+1}{n}\right]=\frac{1}{a+b}\ln(n+1)>\frac{1}{a+b}\ln\left(\left[e^{(a+b)M}\right]+1\right)>M.$$

$$\Rightarrow\lim_{n\to\infty}\left(\frac{1}{a+b}+\frac{1}{2a+b}+\frac{1}{3a+b}+\cdots+\frac{1}{na+b}\right)=+\infty$$

$$5.(2)\lim_{n\to\infty}x_n=0$$

$$\lim_{n\to\infty}\frac{\sqrt{x_n+\sqrt{x_n+\sqrt{x_n}}}}{\sqrt[8]{x_n}}=\lim_{n\to\infty}\sqrt[8]{\frac{\left(x_n+\sqrt{x_n+\sqrt{x_n}}\right)^4}{x_n}}$$

$$=\lim_{n\to\infty}\sqrt[8]{\frac{x_n^4+4x_n^3\sqrt{x_n+\sqrt{x_n}}+6x_n^2\left(\sqrt{x_n+\sqrt{x_n}}\right)^2+4x_n\left(\sqrt{x_n+\sqrt{x_n}}\right)^3+\left(\sqrt{x_n+\sqrt{x_n}}\right)^4}{x_n}}$$

$$=\lim_{n\to\infty}\sqrt[8]{x_n^3+4x_n^2\sqrt{x_n+\sqrt{x_n}}+6x_n\left(\sqrt{x_n+\sqrt{x_n}}\right)^2+4\left(\sqrt{x_n+\sqrt{x_n}}\right)^3+\frac{\left(\sqrt{x_n+\sqrt{x_n}}\right)^4}{x_n}}$$

$$=\sqrt[8]{\lim_{n\to\infty}x_n^3+4\lim_{n\to\infty}x_n^2\sqrt{x_n+\sqrt{x_n}}+6\lim_{n\to\infty}x_n\left(\sqrt{x_n+\sqrt{x_n}}\right)^2+4\lim_{n\to\infty}\left(\sqrt{x_n+\sqrt{x_n}}\right)^3+\lim_{n\to\infty}\frac{\left(\sqrt{x_n+\sqrt{x_n}}\right)^4}{x_n}}$$

$$=\sqrt[8]{\lim_{n\to\infty}x_n^3+4\lim_{n\to\infty}x_n^2\sqrt{x_n+\sqrt{x_n}}+6\lim_{n\to\infty}x_n\left(\sqrt{x_n+\sqrt{x_n}}\right)^2+4\lim_{n\to\infty}\left(\sqrt{x_n+\sqrt{x_n}}\right)^3+\lim_{n\to\infty}\frac{\left(x_n+\sqrt{x_n}\right)^2}{x_n}}$$

$$=\sqrt[8]{\lim_{n\to\infty}x_n^3+4\lim_{n\to\infty}x_n^2\sqrt{x_n+\sqrt{x_n}}+6\lim_{n\to\infty}x_n\left(\sqrt{x_n+\sqrt{x_n}}\right)^2+4\lim_{n\to\infty}\left(\sqrt{x_n+\sqrt{x_n}}\right)^3+\lim_{n\to\infty}x_n+2\lim_{n\to\infty}\sqrt{x_n}+1}$$

$$=1.$$

$$\Rightarrow\sqrt{x_n+\sqrt{x_n+\sqrt{x_n}}}\sim\sqrt[8]{x_n}.$$

$$6.(2)\lim_{n\to\infty}x_n=+\infty\Rightarrow\lim_{n\to\infty}\frac{1}{x_n}=0.$$

$$\lim_{n\to\infty}\frac{\sqrt{x_n+\sqrt{x_n+\sqrt{x_n}}}}{\sqrt{x_n}}=\lim_{n\to\infty}\sqrt{\frac{x_n+\sqrt{x_n+\sqrt{x_n}}}{x_n}}=\lim_{n\to\infty}\sqrt{1+\frac{\sqrt{x_n+\sqrt{x_n}}}{x_n}}=\lim_{n\to\infty}\sqrt{1+\sqrt{\frac{x_n+\sqrt{x_n}}{x_n^2}}}$$

$$=\lim_{n\to\infty}\sqrt{1+\sqrt{\frac{1}{x_n}+\sqrt{\frac{1}{x_n^3}}}}=\sqrt{1+\sqrt{\lim_{n\to\infty}\frac{1}{x_n}+\sqrt{\lim_{n\to\infty}\frac{1}{x_n^3}}}}$$

$$=1.$$

$$\Rightarrow\sqrt{x_n+\sqrt{x_n+\sqrt{x_n}}}\sim\sqrt{x_n}.$$

$1.(1)$

bounded :

obviously : $x_n \ge 0,$

$x_n \le 2 \Rightarrow x_{n+1} = \frac{1}{4}(3x_n + 2) \le 2$

then $x_1 = 0 \le 2 \Rightarrow x_n \le 2, \forall n \in \mathbb{N}$

monotone :

$x_{n+1} = \frac{1}{4}(3x_n + 2) \ge \frac{1}{4}(3x_n + x_n) = x_n$

Hence, $\lim_{n\to\infty} x_n$ *exists*, $\lim_{n\to\infty} x_n \triangleq m.$

$\Rightarrow m = \frac{1}{4}(3m+2) \Rightarrow m = 2.$

$\Rightarrow \lim_{n\to\infty} x_n = 2.$

$1.(2)$

bounded :

obviously : $x_n \ge 0,$

$$x_n \le \frac{1+\sqrt{5}}{2} \Rightarrow x_{n+1} = \frac{1+2x_n}{1+x_n} = 2 - \frac{1}{1+x_n} \le 2 - \frac{1}{1+\frac{1+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2}$$

then $x_1 = 1 \le \frac{1+\sqrt{5}}{2} \Rightarrow x_n \le \frac{1+\sqrt{5}}{2}, \forall n \in \mathbb{N}$

monotone :

$$x_{n+1} = \frac{1+2x_n}{1+x_n} = 2 - \frac{1}{1+x_n} \ge x_n \left(\text{since } \ x_n \le \frac{1+\sqrt{5}}{2}\right)$$

Hence, $\lim_{n\to\infty} x_n$ *exists*, $\lim_{n\to\infty} x_n \triangleq m.$

$\Rightarrow m = \frac{1+2m}{1+m} \Rightarrow m = \frac{1+\sqrt{5}}{2}.$

$\Rightarrow \lim_{n\to\infty} x_n = \frac{1+\sqrt{5}}{2}.$

$1.(5)$

bounded :

obviously : $x_n > 0,$

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{a}{x_n^2}\right) = \frac{1}{3}\left(x_n + x_n + \frac{a}{x_n^2}\right) \ge \sqrt[3]{a}.$$

$\Rightarrow \forall n \in \mathbb{N} - \{1\}, x_n \ge \sqrt[3]{a}.$

monotone :

$$\forall n \in \mathbb{N} - \{1\}, x_{n+1} = \frac{1}{3}\left(2x_n + \frac{a}{x_n^2}\right) \le x_n$$

Hence, $\lim_{n\to\infty} x_n$ *exists*, $\lim_{n\to\infty} x_n \triangleq m.$

$\Rightarrow m = \frac{1}{3}\left(2m + \frac{a}{m^2}\right) \Rightarrow m = \sqrt[3]{a}.$

$\Rightarrow \lim_{n\to\infty} x_n = \sqrt[3]{a}.$

$1.(6)$

bounded :

obviously : $x_n > 0,$

$$x_n = \underbrace{\sqrt{2\sqrt{2\cdots\sqrt{2}}}}_{n\text{个根号}} = \underbrace{\sqrt{2\sqrt{2\cdots\sqrt{2\sqrt{2}}}}}_{n\text{个根号}} < \underbrace{\sqrt{2\sqrt{2\cdots\sqrt{2\cdot 2}}}}_{(n-1)\text{个根号}} = \underbrace{\sqrt{2\sqrt{2\cdots\sqrt{2\cdot 2}}}}_{(n-2)\text{个根号}} = \cdots = 2$$

monotone :

$$x_{n+1} = \underbrace{\sqrt{2\sqrt{2\cdots\sqrt{2}}}}_{(n+1)\text{个根号}} = \sqrt{2\underbrace{\sqrt{2\sqrt{2\cdots\sqrt{2}}}}_{n\text{个根号}}} = \sqrt{2x_n} > \sqrt{x_n \cdot x_n} = x_n$$

Hence, $\lim_{n\to\infty} x_n$ *exists*, $\lim_{n\to\infty} x_n \triangleq m.$

$\Rightarrow m = \sqrt{2m} \Rightarrow m = 2.$

$\Rightarrow \lim_{n\to\infty} x_n = 2.$

3.(1)

$$\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n} \Leftrightarrow \left(1+\frac{1}{n}\right)^{n} < e < \left(1+\frac{1}{n}\right)^{n+1}$$

$$\bullet \because \left(1+\frac{1}{n}\right)^{n} < \left(1+\frac{1}{n+1}\right)^{n+1}, \therefore \left(1+\frac{1}{n}\right)^{n} < \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m} = e$$

$$\bullet Claim: \left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{m(n+1)}\right)^{m(n+1)}$$

$$need\ to\ show: 1+\frac{1}{n} > \left(1+\frac{1}{m(n+1)}\right)^{m}$$

$$proof: 1+\frac{1}{n} = \frac{1}{1-\frac{1}{n+1}} > \frac{1-\left(\frac{1}{n+1}\right)^{m}}{1-\frac{1}{n+1}}$$

$$\left(1+\frac{1}{m(n+1)}\right)^{m} = 1+\frac{C_m^1}{m(n+1)}+\frac{C_m^2}{\left[m(n+1)\right]^2}+\frac{C_m^3}{\left[m(n+1)\right]^3}+\cdots+\frac{C_m^m}{\left[m(n+1)\right]^m}\left(when\ k\ge 2, C_m^k < m^k\right)$$

$$< 1+\frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\cdots+\frac{1}{(n+1)^m} = \frac{1-\left(\frac{1}{n+1}\right)^{m}}{1-\frac{1}{n+1}} < 1+\frac{1}{n}$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{m(n+1)}\right)^{m(n+1)}$$

$fix\ n, let\ m \to +\infty$

$$then\ e = \lim_{m\to\infty}\left(1+\frac{1}{m(n+1)}\right)^{m(n+1)} < \left(1+\frac{1}{n}\right)^{n+1}$$

$$Hence, \frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$$

$3.(2)\ proof:$

$$\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)=\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n=\ln\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]=\ln e=1$$

$3.(3)\ proof:$

$$x_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\left(n=1,2,\cdots\right)$$

$by\ Exercise\ 3.(1):$

$$\frac{1}{n+1}<\ln\left(1+\frac{1}{n}\right)<\frac{1}{n}$$

$$\Rightarrow \ln\frac{n+1}{n}<\frac{1}{n}<\ln\frac{n}{n-1},n\geq 2$$

$$\ln\left(n+1\right)=\sum_{k=1}^{n}\ln\frac{k+1}{k}<\sum_{k=1}^{n}\frac{1}{k}<1+\sum_{k=2}^{n}\ln\frac{k}{k-1}=1+\ln n$$

$$\Rightarrow \ln\left(n+1\right)-\ln n<x_n=\sum_{k=1}^{n}\frac{1}{k}-\ln n<1+\ln n-\ln n$$

$$\Leftrightarrow \ln\frac{n+1}{n}<x_n=\sum_{k=1}^{n}\frac{1}{k}-\ln n<1$$

$$0=\lim_{n\to\infty}\ln\frac{n+1}{n}\leq\lim_{n\to\infty}x_n\leq 1.$$

$$\Rightarrow \lim_{n\to\infty}x_n\ exists.$$

$3.(4)\ proof:$

$according\ to\ Exercise\ 3.(3):$

$$\lim_{n\to\infty}x_n=c$$

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)=\lim_{n\to\infty}\left(\sum_{k=1}^{2n}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{k}\right)=\lim_{n\to\infty}\left[\left(\ln 2n+c\right)-\left(\ln n+c\right)\right]=\ln 2.$$

$6.proof:$

(A,B) *is a Dedeking cut of* $\mathbb{R}$.

select $a_1 \in A, b_1 \in B \Rightarrow a_1 < b_1$

for $k \geq 1, k \in \mathbb{N}$,

$$\frac{a_k + b_k}{2} \in A \cup B = \mathbb{R} \Rightarrow \frac{a_k + b_k}{2} \in A \text{ or } \frac{a_k + b_k}{2} \in B$$

$$\text{if } \frac{a_k + b_k}{2} \in A, \text{let } a_{k+1} = \frac{a_k + b_k}{2}, b_{k+1} = b_k \Rightarrow [a_{k+1}, b_{k+1}] \subset [a_k, b_k]$$

$$\text{if } \frac{a_k + b_k}{2} \in B, \text{let } b_{k+1} = \frac{a_k + b_k}{2}, a_{k+1} = a_k \Rightarrow [a_{k+1}, b_{k+1}] \subset [a_k, b_k]$$

then

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_k, b_k] \supseteq [a_{k+1}, b_{k+1}] \supseteq \cdots$$

$$\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} \frac{1}{2^{n-1}}(b_1 - a_1) = 0$$

$$\therefore \exists! c \in \mathbb{R}, \forall n \in \mathbb{N}, s.t. c \in [a_n, b_n], \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c. (a_n \in A, b_n \in B)$$

Claim: $\forall x \in A, y \in B, x \leq c \leq y$.

$$\text{if } \exists x \in A, x > c, \text{then } b_n \in B \Rightarrow b_n - c > x - c > 0 \Rightarrow \lim_{n\to\infty} b_n \neq c \Rightarrow \forall x \in A, x \leq c.$$

Similarly, $\forall y \in B, c \leq y$.

Hence, $\forall x \in A, y \in B, x \leq c \leq y$.

$7.proof$:

$S \in \mathbb{R}$ is upper bounded,and $S \neq \varnothing$.

$select\ a_1 =$ the upper bound of $S, b_1 \in S$

$for\ k \geq 1, k \in \mathbb{N},$

$$\frac{a_k + b_k}{2} \in \mathbb{R} \Rightarrow \frac{a_k + b_k}{2} \in S\ or \frac{a_k + b_k}{2} \in \mathbb{R} - S$$

$$if \frac{a_k + b_k}{2} \in S, let\ a_{k+1} = \frac{a_k + b_k}{2}, b_{k+1} = b_k \Rightarrow [a_{k+1}, b_{k+1}] \subset [a_k, b_k]$$

$$if \frac{a_k + b_k}{2} \in \mathbb{R} - S, let\ b_{k+1} = \frac{a_k + b_k}{2}, a_{k+1} = a_k \Rightarrow [a_{k+1}, b_{k+1}] \subset [a_k, b_k]$$

$then$

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_k, b_k] \supseteq [a_{k+1}, b_{k+1}] \supseteq \cdots$$

$$\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} \frac{1}{2^{n-1}}(b_1 - a_1) = 0$$

$$\therefore \exists! c \in \mathbb{R}, \forall n \in \mathbb{N}, s.t. c \in [a_n, b_n], \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c.(a_n \in S, b_n \in \mathbb{R} - S)$$

$Claim : c$ is the superium of S.

$$need\ to\ show : \begin{cases} \forall x \in S, x \leq c \\ \forall \varepsilon > 0, \exists x_\varepsilon \in S, s.t. x_\varepsilon > c - \varepsilon \end{cases}$$

$\bullet if\ \exists x \in S, x > c :$

$$b_n \in \mathbb{R} - S \Rightarrow b_n - c > x - c > 0 \Rightarrow \lim_{n\to\infty} b_n \neq c$$

$$\Rightarrow \forall x \in S, x \leq c$$

$\bullet \exists \varepsilon > 0, \forall x \in S, s.t. x \leq c - \varepsilon :$

$$a_n \in S \Rightarrow c - a_n \geq \varepsilon > 0 \Rightarrow \lim_{n\to\infty} a_n \neq c.$$

$$\Rightarrow \forall \varepsilon > 0, \exists x_\varepsilon \in S, s.t. x_\varepsilon > c - \varepsilon.$$

$Hence, c$ is the superium of $S, \sup S\ exists$

$Similarly, if\ S \in \mathbb{R}$ is lower bounded,and $S \neq \varnothing, then \inf S\ exists.$

10/10 作业

9.(1)

$$x_{n+1}=\frac{1}{3}(2x_n+1)\Leftrightarrow x_{n+1}-1=\frac{2}{3}(x_n-1)\Rightarrow\begin{cases}|x_{n+1}-1|=0+\frac{2}{3}|x_n-1|\\ \frac{2}{3}\in(0,1)\end{cases}\Rightarrow\lim_{n\to\infty}x_n-1=0\Rightarrow\lim_{n\to\infty}x_n=1$$

9.(3)

$$x_1=2,x_{n+1}=2+\frac{1}{x_n}=\frac{2x_n+1}{x_n}.$$

$obviously, x_n\geq 2.$

$$x_{n+1}-\frac{\sqrt{5}+1}{2}=\frac{2x_n+1}{x_n}-\frac{\sqrt{5}+1}{2}=\frac{2x_n+1-\frac{\sqrt{5}+1}{2}x_n}{x_n}=-\frac{\frac{\sqrt{5}-1}{2}x_n-1}{x_n}=-\frac{\sqrt{5}-1}{2x_n}\left(x_n-\frac{\sqrt{5}+1}{2}\right)$$

$$\begin{cases}\left|x_{n+1}-\frac{\sqrt{5}+1}{2}\right|=0+\left|\frac{\sqrt{5}-1}{2x_n}\right|\left|x_n-\frac{\sqrt{5}+1}{2}\right|\\ \left|\frac{\sqrt{5}-1}{2x_n}\right|\leq\frac{\sqrt{5}-1}{4},\left|\frac{\sqrt{5}-1}{2x_n}\right|\in(0,1)\end{cases}\Rightarrow\lim_{n\to\infty}x_n-\frac{\sqrt{5}+1}{2}=0\Rightarrow\lim_{n\to\infty}x_n=\frac{\sqrt{5}+1}{2}.$$

$10.(1)\exists 0<\lambda<1, s.t.|x_{n+1}-x_n|\leq\lambda|x_n-x_{n-1}|\leq\cdots\leq\lambda^{n-1}|x_2-x_1|$

$$\forall\varepsilon>0,\exists N=\max\left\{\left[\log_\lambda\frac{(1-\lambda)\varepsilon}{|x_2-x_1|}\right]+1,1\right\}\in\mathbb{N}, s.t.|x_m-x_n|<\varepsilon,\forall m>n>N$$

$$|x_m-x_n|=|(x_m-x_{m-1})+(x_{m-1}-x_{m-2})+\cdots+(x_{n+1}-x_n)|$$

$$\leq|x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n|\leq\lambda^{m-2}|x_2-x_1|+\lambda^{m-3}|x_2-x_1|+\cdots+\lambda^{n-1}|x_2-x_1|$$

$$=\frac{\lambda^{n-1}-\lambda^{m-1}}{1-\lambda}|x_2-x_1|<\frac{\lambda^{n-1}}{1-\lambda}|x_2-x_1|<\varepsilon$$

$$\overset{Cauchy\ convergrnce\ criterion}{\Rightarrow}\lim_{n\to\infty}x_n\ exists.$$

$$10.(2)x_1=1,x_{n+1}=\frac{2+x_n}{1+x_n}=1+\frac{1}{1+x_n}\geq 1$$

$$|x_{n+1}-x_n|=\left|\frac{1}{1+x_n}-\frac{1}{1+x_{n-1}}\right|=\left|\frac{x_{n-1}-x_n}{(1+x_n)(1+x_{n-1})}\right|\leq\frac{1}{4}|x_n-x_{n-1}|\Rightarrow\{x_n\}converge$$

$$\lim_{n\to\infty}x_n\triangleq m\Rightarrow m=\frac{2+m}{1+m}\Rightarrow m=\sqrt{2}\Rightarrow\lim_{n\to\infty}x_n=\sqrt{2}$$

$$10.(3)x_1=1,x_{n+1}=\sqrt{2+x_n}\geq\sqrt{2}$$

$$|x_{n+1}-x_n|=\left|\sqrt{2+x_n}-\sqrt{2+x_{n-1}}\right|=\left|\frac{x_n-x_{n-1}}{\sqrt{2+x_n}+\sqrt{2+x_{n-1}}}\right|\leq\left|\frac{x_n-x_{n-1}}{2\sqrt{2+\sqrt{2}}}\right|=\frac{1}{2\sqrt{2+\sqrt{2}}}|x_n-x_{n-1}|\Rightarrow\{x_n\}converge$$

$$\lim_{n\to\infty}x_n\triangleq m\geq\sqrt{2}\Rightarrow m=\sqrt{2+m}\Rightarrow m=2\Rightarrow\lim_{n\to\infty}x_n=2$$

$$10.(4)x_1=1,x_{n+1}=1+\frac{1}{x_n}>1$$

$$\begin{cases}|x_{n+1}-x_n|=\left|\frac{1}{x_n}-\frac{1}{x_{n-1}}\right|=\left|\frac{x_{n-1}-x_n}{x_nx_{n-1}}\right|\leq\frac{1}{x_nx_{n-1}}|x_n-x_{n-1}|\\ 0<\frac{1}{x_nx_{n-1}}<1\end{cases}\Rightarrow\{x_n\}converge$$

$$\lim_{n\to\infty}x_n\triangleq m\Rightarrow m=1+\frac{1}{m}\Rightarrow m=\frac{1+\sqrt{5}}{2}\Rightarrow\lim_{n\to\infty}x_n=\frac{1+\sqrt{5}}{2}$$

$12.proof:$

$\{x_n\}$ is bounded and monotonic.Without loss of generality: we assume that $\{x_n\}$ monotonically increases.

$\overset{\text{bounded}}{\Rightarrow} \exists\{n_1, n_2, \cdots, n_k, \cdots\} \subseteq \mathbb{N}, s.t. \{x_{n_k}\}$ converge at a point denoted by "p".

$\Rightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n_k > N, \left|x_{n_k} - p\right| < \varepsilon.$

$\because \{x_n\}$ is monotonically increasing

$\therefore \forall n_k > N, \forall t \in [n_k, n_{k+1}) \cap \mathbb{N}, s.t. x_{n_k} \le x_t < x_{n_{k+1}}$

$\Rightarrow |x_t - p| \le \min\left\{\left|x_{n_k} - p\right|, \left|x_{n_{k+1}} - p\right|\right\} < \varepsilon$

$\inf \bigcup_{n_k > N} \left([n_k, n_{k+1}) \cap \mathbb{N}\right) \triangleq M$

$$\Rightarrow \bigcup_{n_k > N} \left([n_k, n_{k+1}) \cap \mathbb{N}\right) = \bigcup_{n_k > N} \{t : t \in [n_k, n_{k+1}) \cap \mathbb{N}\} = \left\{t : t \in \bigcup_{n_k > N} \left([n_k, n_{k+1}) \cap \mathbb{N}\right)\right\}$$

$$= \left\{t : t \in \left(\bigcup_{n_k > N} [n_k, n_{k+1})\right) \cap \mathbb{N}\right\} = \{t : t \in [M, +\infty) \cap \mathbb{N}\} = [M, +\infty) \cap \mathbb{N}$$

$$\because \bigcup_{k=1}^{\infty} \left([n_k, n_{k+1}) \cap \mathbb{N}\right) = \bigcup_{k=1}^{\infty} \{t : t \in [n_k, n_{k+1}) \cap \mathbb{N}\} = \left\{t : t \in \bigcup_{k=1}^{\infty} \left([n_k, n_{k+1}) \cap \mathbb{N}\right)\right\} = \left\{t : t \in \left(\bigcup_{k=1}^{\infty} [n_k, n_{k+1})\right) \cap \mathbb{N}\right\} = \{t : t \in \mathbb{N}\} = \mathbb{N}$$

$$\therefore \forall t \in \bigcup_{n_k > N} \left([n_k, n_{k+1}) \cap \mathbb{N}\right) = [M, +\infty) \cap \mathbb{N} = \{M, M+1, M+2, \cdots\}, s.t. |x_t - p| < \varepsilon$$

$i.e. \forall \varepsilon > 0, \exists M \in \mathbb{N}, s.t. |x_n - p| < \varepsilon, \forall n > M.$

$\Rightarrow \{x_n\}$ converges.

$13.(2)\, proof:$

$$\forall \varepsilon > 0, \exists N = \left[\frac{1}{\varepsilon}\right] + 1 \in \mathbb{N}, s.t. \forall m > n > N, |x_m - x_n| = \left|\sum_{k=1}^{m} \frac{\cos a_k}{k(k+1)} - \sum_{k=1}^{n} \frac{\cos a_k}{k(k+1)}\right| = \left|\sum_{k=n}^{m} \frac{\cos a_k}{k(k+1)}\right|$$

$$\le \sum_{k=n}^{m} \frac{|\cos a_k|}{k(k+1)} \le \sum_{k=n}^{m} \frac{1}{k(k+1)} = \sum_{k=n}^{m} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{n} - \frac{1}{m+1} < \frac{1}{n} < \frac{1}{N} < \varepsilon \Rightarrow \{x_n\}\, converges$$

$13.(3)\, proof:$

$$\forall \varepsilon > 0, \exists N = \max\left\{\left[\log_{10} \frac{1}{\varepsilon}\right] + 3, 3\right\} \in \mathbb{N}, s.t. \forall m > n > N, |x_m - x_n| = \left|\sum_{k=1}^{m} \frac{a_k}{10^k} - \sum_{k=1}^{n} \frac{a_k}{10^k}\right| = \left|\sum_{k=n}^{m} \frac{a_k}{10^k}\right| \le \sum_{k=n}^{m} \frac{|a_k|}{10^k}$$

$$< \sum_{k=n}^{m} \frac{10}{10^k} = \sum_{k=n}^{m} \frac{1}{10^{k-1}} = \frac{\frac{1}{10^{n-1}} - \frac{1}{10^m}}{1 - \frac{1}{10}} < \frac{\frac{1}{10^{n-1}}}{1 - \frac{1}{10}} = \frac{10}{9} \frac{1}{10^{n-1}} < 10 \cdot \frac{1}{10^{n-1}} = \frac{1}{10^{n-2}} < \varepsilon \Rightarrow \{x_n\}\, converges$$

Exercise 2.5

$1.(1)\; x_n = (-1)^n \sqrt[n]{n} + \frac{1}{\sqrt[n]{n}}$

$obviously, \overline{x_n} = \overline{(-1)^n \sqrt[n]{n} + \frac{1}{\sqrt[n]{n}}} = \sqrt[n]{n} + \frac{1}{\sqrt[n]{n}}, \underline{x_n} = \underline{(-1)^n \sqrt[n]{n} + \frac{1}{\sqrt[n]{n}}} = -\sqrt[n]{n} + \frac{1}{\sqrt[n]{n}}$

$\text{since} \lim_{n\to\infty} \sqrt[n]{n} = 1, then:$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \overline{x_n} = \lim_{n\to\infty} \sqrt[n]{n} + \frac{1}{\sqrt[n]{n}} = \lim_{n\to\infty} \sqrt[n]{n} + \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = \lim_{n\to\infty} \sqrt[n]{n} + \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{n}} = 1+1=2$$

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \underline{x_n} = \lim_{n\to\infty} -\sqrt[n]{n} + \frac{1}{\sqrt[n]{n}} = \lim_{n\to\infty} -\sqrt[n]{n} + \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = \lim_{n\to\infty} -\sqrt[n]{n} + \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{n}} = -1+1=0$$

$1.(3)\; x_n = \sqrt[n]{1+2^{n(-1)^n}},$

$obviously, \overline{x_n} = \overline{\sqrt[n]{1+2^{n(-1)^n}}} = \sqrt[n]{1+2^n}, \underline{x_n} = \underline{\sqrt[n]{1+2^{n(-1)^n}}} = \sqrt[n]{1+2^{-n}}$

$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \overline{x_n}, \liminf_{n\to\infty} x_n = \lim_{n\to\infty} \underline{x_n}$

$$\because \ln 2 = \lim_{n\to\infty} \frac{1}{n} \ln 2^n \le \lim_{n\to\infty} \frac{1}{n} \ln\left(1+2^n\right) \le \lim_{n\to\infty} \frac{1}{n} \ln\left(2^{n+1}\right) = \ln 2$$

$$\therefore \limsup_{n\to\infty} x_n = \lim_{n\to\infty} \overline{x_n} = \lim_{n\to\infty} \sqrt[n]{1+2^n} = e^{\lim\limits_{n\to\infty} \frac{1}{n} \ln\left(1+2^n\right)} = e^{\ln 2} = 2$$

$$\because -\ln 2 = \lim_{n\to\infty} \frac{1}{n} \ln 2^{-n} \le \lim_{n\to\infty} \frac{1}{n} \ln\left(1+2^{-n}\right) \le \lim_{n\to\infty} \frac{1}{n} \ln\left(2^{-n+1}\right) = -\ln 2$$

$$\therefore \liminf_{n\to\infty} x_n = \lim_{n\to\infty} \underline{x_n} = \lim_{n\to\infty} \sqrt[n]{1+2^{-n}} = e^{\lim\limits_{n\to\infty} \frac{1}{n} \ln\left(1+2^{-n}\right)} = e^{-\ln 2} = \frac{1}{2}$$

$2.(1)$:

$\forall \varepsilon > 0,$

by Theorem2.5.1:

$$\begin{cases} \exists \left\{x_{s_k}^m\right\}_{k\in\mathbb{N}} \subset \left\{x_n^m\right\}, s.t. \exists N_1 \in \mathbb{N}, \forall k > N_1, \left|x_{s_k}^m - \liminf\limits_{n\to\infty} x_n^m\right| < \dfrac{\varepsilon}{3} \\ \exists \left\{x_{t_k}\right\}_{k\in\mathbb{N}} \subset \left\{x_n\right\}, s.t. \exists N_2 \in \mathbb{N}, \forall k > N_2, \left|x_{t_k} - \liminf\limits_{n\to\infty} x_n\right| < \varepsilon_k = \min\left\{\dfrac{1}{2}, \dfrac{\varepsilon}{3m!} \cdot \dfrac{1}{\left|M^{m-1} + M^{m-2} + \cdots + 1\right|}\right\} \end{cases}$$

$\because \left\{x_n\right\}$ is bounded$\left(\sup\limits_{n\in\mathbb{N}} x_n \triangleq M\right)$

by Bolzano-Weiestrass Theorem:

$\left\{x_n\right\} converges.$

by Cauchy Convergence Criterion:

$\exists N_3 \in \mathbb{N}, s.t. \forall m > n > N_3, \left|x_m - x_n\right| < \dfrac{\varepsilon}{3}$

$k > N_3 \Rightarrow s_k, t_k > N_3\ , then \forall m > n > N_3, \left|x_{s_k}^m - x_{t_k}^m\right| < \dfrac{\varepsilon}{3}$

$\exists N = \max\left\{N_1, N_2, N_3\right\}, s.t. \forall n > N,$

$$\left|\liminf_{n\to\infty} x_n^m - \left(\liminf_{n\to\infty} x_n\right)^m\right| = \left|\liminf_{n\to\infty} x_n^m - x_{s_k}^m + x_{s_k}^m - \left(\liminf_{n\to\infty} x_n\right)^m\right|$$

$$\le \left|\liminf_{n\to\infty} x_n^m - x_{s_k}^m\right| + \left|x_{s_k}^m - \left(\liminf_{n\to\infty} x_n\right)^m\right| < \frac{\varepsilon}{3} + \left|x_{s_k}^m - x_{t_k}^m + x_{t_k}^m - \left(\liminf_{n\to\infty} x_n\right)^m\right|$$

$$\le \frac{\varepsilon}{3} + \left|x_{s_k}^m - x_{t_k}^m\right| + \left|x_{t_k}^m - \left(\liminf_{n\to\infty} x_n\right)^m\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|x_{t_k}^m - \left(\liminf_{n\to\infty} x_n\right)^m\right|$$

$$= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|x_{t_k}^m - \left(\liminf_{n\to\infty} x_n\right)^m\right| \le \frac{\varepsilon}{3} + \varepsilon + \left|x_{t_k}^m - \left(x_{t_k} + \varepsilon_k\right)^m\right|$$

$$= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|x_{t_k}^m - \left(x_{t_k}^m + C_m^1 \varepsilon_k x_{t_k}^{m-1} + C_m^2 \varepsilon_k^2 x_{t_k}^{m-2} + \cdots + C_m^m \varepsilon_k^m x_{t_k}\right)\right|$$

$$= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|C_m^1 \varepsilon_k x_{t_k}^{m-1} + C_m^2 \varepsilon_k^2 x_{t_k}^{m-2} + \cdots + C_m^m \varepsilon_k^m\right|$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|C_m^1 x_{t_k}^{m-1} + C_m^2 x_{t_k}^{m-2} + \cdots + C_m^m\right| \varepsilon_k \left(C_m^n = \frac{m!}{n!(m-n)!} \le m!\right)$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|x_{t_k}^{m-1} + x_{t_k}^{m-2} + \cdots + 1\right| m! \varepsilon_k$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|M^{m-1} + M^{m-2} + \cdots + 1\right| m! \varepsilon_k = \varepsilon$$

$$\Rightarrow \forall \varepsilon > 0, \exists N = \max\left\{N_1, N_2, N_3\right\} \in \mathbb{N}, s.t. \forall n > N, \left|\liminf_{n\to\infty} x_n^m - \left(\liminf_{n\to\infty} x_n\right)^m\right| < \varepsilon$$

$$\Rightarrow \liminf_{n\to\infty} x_n^m = \left(\liminf_{n\to\infty} x_n\right)^m$$

2.(1):

$\forall \varepsilon > 0,$

by Theorem2.5.1:

$$\begin{cases} \exists \left\{x_{s_k}^m\right\}_{k\in\mathbb{N}} \subset \left\{x_n^m\right\}, s.t. \exists N_1 \in \mathbb{N}, \forall k > N_1, \left|x_{s_k}^m - \limsup_{n\to\infty} x_n^m\right| < \frac{\varepsilon}{3} \\ \exists \left\{x_{t_k}\right\}_{k\in\mathbb{N}} \subset \left\{x_n\right\}, s.t. \exists N_2 \in \mathbb{N}, \forall k > N_2, \left|x_{t_k} - \limsup_{n\to\infty} x_n\right| < \varepsilon_k = \min\left\{\frac{1}{2}, \frac{\varepsilon}{3m!} \cdot \frac{1}{\left|M^{m-1} + M^{m-2} + \cdots + 1\right|}\right\} \end{cases}$$

$\because \left\{x_n\right\}$ is bounded$\left(\sup_{n\in\mathbb{N}} x_n \triangleq M\right)$

by Bolzano-Weiestrass Theorem:

$\left\{x_n\right\}$ *converges.*

by Cauchy Convergence Criterion:

$\exists N_3 \in \mathbb{N}, s.t. \forall m > n > N_3, \left|x_m - x_n\right| < \dfrac{\varepsilon}{3}$

$k > N_3 \Rightarrow s_k, t_k > N_3$ $, then \forall m > n > N_3, \left|x_{s_k}^m - x_{t_k}^m\right| < \dfrac{\varepsilon}{3}$

$\exists N = \max\left\{N_1, N_2, N_3\right\}, s.t. \forall n > N,$

$$\left|\limsup_{n\to\infty} x_n^m - \left(\limsup_{n\to\infty} x_n\right)^m\right| = \left|\limsup_{n\to\infty} x_n^m - x_{s_k}^m + x_{s_k}^m - \left(\limsup_{n\to\infty} x_n\right)^m\right|$$

$$\le \left|\limsup_{n\to\infty} x_n^m - x_{s_k}^m\right| + \left|x_{s_k}^m - \left(\limsup_{n\to\infty} x_n\right)^m\right| < \frac{\varepsilon}{3} + \left|x_{s_k}^m - x_{t_k}^m + x_{t_k}^m - \left(\limsup_{n\to\infty} x_n\right)^m\right|$$

$$\le \frac{\varepsilon}{3} + \left|x_{s_k}^m - x_{t_k}^m\right| + \left|x_{t_k}^m - \left(\limsup_{n\to\infty} x_n\right)^m\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|x_{t_k}^m - \left(\limsup_{n\to\infty} x_n\right)^m\right|$$

$$= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|x_{t_k}^m - \left(\limsup_{n\to\infty} x_n\right)^m\right| \le \frac{\varepsilon}{3} + \varepsilon + \left|x_{t_k}^m - \left(x_{t_k} + \varepsilon_k\right)^m\right|$$

$$= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|x_{t_k}^m - \left(x_{t_k}^m + C_m^1 \varepsilon_k x_{t_k}^{m-1} + C_m^2 \varepsilon_k^2 x_{t_k}^{m-2} + \cdots + C_m^m \varepsilon_k^m x_{t_k}\right)\right|$$

$$= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|C_m^1 \varepsilon_k x_{t_k}^{m-1} + C_m^2 \varepsilon_k^2 x_{t_k}^{m-2} + \cdots + C_m^m \varepsilon_k^m\right|$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|C_m^1 x_{t_k}^{m-1} + C_m^2 x_{t_k}^{m-2} + \cdots + C_m^m\right| \varepsilon_k \left(C_m^n = \frac{m!}{n!(m-n)!} \le m!\right)$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|x_{t_k}^{m-1} + x_{t_k}^{m-2} + \cdots + 1\right| m! \varepsilon_k$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|M^{m-1} + M^{m-2} + \cdots + 1\right| m! \varepsilon_k = \varepsilon$$

$$\Rightarrow \forall \varepsilon > 0, \exists N = \max\left\{N_1, N_2, N_3\right\} \in \mathbb{N}, s.t. \forall n > N, \left|\limsup_{n\to\infty} x_n^m - \left(\limsup_{n\to\infty} x_n\right)^m\right| < \varepsilon$$

$$\Rightarrow \limsup_{n\to\infty} x_n^m = \left(\limsup_{n\to\infty} x_n\right)^m$$

2.(2):

$\forall \varepsilon > 0,$

by Theorem2.5.1:

$$\begin{cases} \exists \left\{ x_{s_k}^m \right\}_{k\in\mathbb{N}} \subset \left\{ x_n^m \right\}, s.t. \exists N_1 \in \mathbb{N}, \forall k > N_1, \left| \dfrac{1}{x_{s_k}} - \liminf\limits_{n\to\infty} \dfrac{1}{x_n} \right| < \dfrac{\varepsilon}{3} \\ \exists \left\{ x_{t_k} \right\}_{k\in\mathbb{N}} \subset \left\{ x_n \right\}, s.t. \exists N_2 \in \mathbb{N}, \forall k > N_2, \left| x_{t_k} - \limsup\limits_{n\to\infty} x_n \right| < \left(\inf\limits_{n\in\mathbb{N}} x_n \right)^2 \dfrac{\varepsilon}{3} \end{cases}$$

$\because \left\{ x_n \right\}$ is bounded $\left(x_n > 0 \right)$

$\therefore \left\{ \dfrac{1}{x_n} \right\}$ is bounded $\left(\sup\limits_{n\in\mathbb{N}} \dfrac{1}{x_n} \triangleq M \right)$

$\Rightarrow M = \dfrac{1}{\inf\limits_{n\in\mathbb{N}} x_n} > 0$

by Bolzano-Weiestrass Theorem:

$\left\{ x_n \right\}$ *converges.*

by Cauchy Convergence Criterion:

$\exists N_3 \in \mathbb{N}, s.t. \forall m > n > N_3, \left| \dfrac{1}{x_m} - \dfrac{1}{x_n} \right| < \dfrac{\varepsilon}{3}$

$k > N_3 \Rightarrow s_k, t_k > N_3$ $, then \forall m > n > N_3, \left| \dfrac{1}{x_{s_k}} - \dfrac{1}{x_{t_k}} \right| < \dfrac{\varepsilon}{3}$

$\exists N = \max\left\{ N_1, N_2, N_3 \right\}, s.t. \forall n > N,$

$$\left| \liminf_{n\to\infty} \frac{1}{x_n} - \limsup_{n\to\infty} x_n \right| = \left| \liminf_{n\to\infty} \frac{1}{x_n} - \frac{1}{x_{s_k}} + \frac{1}{x_{s_k}} - \frac{1}{x_{t_k}} + \frac{1}{x_{t_k}} - \frac{1}{\limsup\limits_{n\to\infty} x_n} \right|$$

$$\le \left| \liminf_{n\to\infty} \frac{1}{x_n} - \frac{1}{x_{s_k}} \right| + \left| \frac{1}{x_{s_k}} - \frac{1}{x_{t_k}} \right| + \left| \frac{1}{x_{t_k}} - \frac{1}{\limsup\limits_{n\to\infty} x_n} \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left| \frac{\limsup\limits_{n\to\infty} x_n - x_{t_k}}{x_{t_k} \limsup\limits_{n\to\infty} x_n} \right| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left| \frac{\limsup\limits_{n\to\infty} x_n - x_{t_k}}{\inf\limits_{n\in\mathbb{N}} x_n \cdot \inf\limits_{n\in\mathbb{N}} x_n} \right| = \varepsilon$$

$\Rightarrow \forall \varepsilon > 0, \exists N = \max\left\{ N_1, N_2, N_3 \right\} \in \mathbb{N}, s.t. \forall n > N, \left| \liminf\limits_{n\to\infty} \dfrac{1}{x_n} - \limsup\limits_{n\to\infty} x_n \right| < \varepsilon$

$\Rightarrow \liminf\limits_{n\to\infty} \dfrac{1}{x_n} = \limsup\limits_{n\to\infty} x_n$

$\because \left\{ x_n \right\}$ is bounded $\left(x_n > 0 \right)$

$\therefore \left\{ \dfrac{1}{x_n} \right\}$ is bounded, $\inf\limits_{n\in\mathbb{N}} \dfrac{1}{x_n} > 0$

replace x_n by $\dfrac{1}{x_n} \Rightarrow \liminf\limits_{n\to\infty} x_n = \limsup\limits_{n\to\infty} \dfrac{1}{x_n}$

$3.(1)\ x_{n+1}=\frac{1}{3}\left(x_n+2\right)$

$\limsup\limits_{n\to\infty} x_{n+1}=\frac{1}{3}\left(\limsup\limits_{n\to\infty} x_n+2\right)\Rightarrow \limsup\limits_{n\to\infty} x_n=1$

$\liminf\limits_{n\to\infty} x_{n+1}=\frac{1}{3}\left(\liminf\limits_{n\to\infty} x_n+2\right)\Rightarrow \liminf\limits_{n\to\infty} x_n=1$

$3.(3)\ x_1=\sqrt{3},x_2=\sqrt{3\sqrt{3}},x_3=\sqrt{3\sqrt{3\sqrt{3}}},\cdots,x_n=\underbrace{\sqrt{3\sqrt{3\cdots\sqrt{3}}}}_{n\text{个根号}}$

$$x_n=\underbrace{\sqrt{3\sqrt{3\cdots\sqrt{3}}}}_{n\text{个根号}}=\sqrt{3\underbrace{\sqrt{3\sqrt{3\cdots\sqrt{3}}}}_{(n-1)\text{个根号}}}=\sqrt{3x_{n-1}}$$

$\limsup\limits_{n\to\infty} x_n=\limsup\limits_{n\to\infty}\sqrt{3x_{n-1}}=\sqrt{3}\limsup\limits_{n\to\infty}\sqrt{x_{n-1}}=\sqrt{3}\sqrt{\limsup\limits_{n\to\infty} x_{n-1}}\Rightarrow \limsup\limits_{n\to\infty} x_n=3$

$\liminf\limits_{n\to\infty} x_n=\liminf\limits_{n\to\infty}\sqrt{3x_{n-1}}=\sqrt{3}\liminf\limits_{n\to\infty}\sqrt{x_{n-1}}=\sqrt{3}\sqrt{\liminf\limits_{n\to\infty} x_{n-1}}\Rightarrow \liminf\limits_{n\to\infty} x_n=3$

$4.(1)\ x_1=2,x_{n+1}=\frac{3+2x_n}{2+3x_n}=\frac{2}{3}+\frac{\frac{5}{3}}{2+3x_n}$

$L\triangleq\limsup\limits_{n\to\infty} x_n, l\triangleq\liminf\limits_{n\to\infty} x_n$

$$\begin{cases}\limsup\limits_{n\to\infty} x_{n+1}=\limsup\limits_{n\to\infty}\left(\frac{2}{3}+\frac{\frac{5}{3}}{2+3x_n}\right)=\frac{2}{3}+\frac{\frac{5}{3}}{2+3\liminf\limits_{n\to\infty} x_n}\Leftrightarrow L=\frac{2}{3}+\frac{\frac{5}{3}}{2+3l}\\ \liminf\limits_{n\to\infty} x_{n+1}=\liminf\limits_{n\to\infty}\left(\frac{2}{3}+\frac{\frac{5}{3}}{2+3x_n}\right)=\frac{2}{3}+\frac{\frac{5}{3}}{2+3\limsup\limits_{n\to\infty} x_n}\Leftrightarrow l=\frac{2}{3}+\frac{\frac{5}{3}}{2+3L}\end{cases}$$

$\Rightarrow L=l=1\Rightarrow \lim\limits_{n\to\infty} x_n=1$

$4.(2)\ x_1=3,x_2=3+\frac{1}{3},x_3=3+\frac{1}{3+\frac{1}{3}},\cdots,x_{n+1}=3+\frac{1}{x_n}$

$L\triangleq\limsup\limits_{n\to\infty} x_n, l\triangleq\liminf\limits_{n\to\infty} x_n, L,l\geq 3$

$$\begin{cases}\limsup\limits_{n\to\infty} x_{n+1}=\limsup\limits_{n\to\infty}\left(3+\frac{1}{x_n}\right)=3+\frac{1}{\liminf\limits_{n\to\infty} x_n}\Leftrightarrow L=3+\frac{1}{l}\\ \liminf\limits_{n\to\infty} x_{n+1}=\liminf\limits_{n\to\infty}\left(3+\frac{1}{x_n}\right)=3+\frac{1}{\limsup\limits_{n\to\infty} x_n}\Leftrightarrow l=3+\frac{1}{L}\end{cases}$$

$\Rightarrow L=l=\frac{3+\sqrt{13}}{2}\Rightarrow \lim\limits_{n\to\infty} x_n=\frac{3+\sqrt{13}}{2}$

$$5.(1)\ proof: 0<x_1<y_1, x_{n+1}=\sqrt{x_n y_n}, y_{n+1}=\frac{x_n+y_n}{2}$$

$$x_{n+1}=\sqrt{x_n y_n}\le\frac{x_n+y_n}{2}=y_{n+1}\Rightarrow\begin{cases}y_{n+1}=\dfrac{x_n+y_n}{2}\le y_n\\ x_{n+1}=\sqrt{x_n y_n}\ge x_n\end{cases}$$

$$\Rightarrow\begin{cases}x_1\le\cdots\le x_n\le x_{n+1}\le\cdots\\ y_1\ge y_2\ge\cdots\ge y_{n+1}\ge\cdots\end{cases}\Rightarrow\forall n\in\mathbb{N}, 0<x_1\le x_n\le y_1, 0<x_1\le y_n\le y_1.$$

By Bolzano – Weierstrass Theorem :

$\lim_{n\to\infty}x_n, \lim_{n\to\infty}y_n$ *exists*

$$\limsup_{n\to\infty}y_{n+1}=\limsup_{n\to\infty}\frac{x_n+y_n}{2}=\frac{\limsup_{n\to\infty}x_n+\limsup_{n\to\infty}y_n}{2}\Rightarrow\limsup_{n\to\infty}x_n=\limsup_{n\to\infty}y_n$$

$$\liminf_{n\to\infty}y_{n+1}=\liminf_{n\to\infty}\frac{x_n+y_n}{2}=\frac{\liminf_{n\to\infty}x_n+\liminf_{n\to\infty}y_n}{2}\Rightarrow\liminf_{n\to\infty}x_n=\liminf_{n\to\infty}y_n$$

$$\frac{y_{n+1}}{x_{n+1}}=\frac{x_n+y_n}{2\sqrt{x_n y_n}}=\frac{1}{2}\sqrt{\frac{x_n}{y_n}}+\frac{1}{2}\sqrt{\frac{y_n}{x_n}}, L\triangleq\limsup_{n\to\infty}\frac{x_n}{y_n}, l\triangleq\liminf_{n\to\infty}\frac{x_n}{y_n}, L,l>0$$

$$\Rightarrow\begin{cases}\limsup\limits_{n\to\infty}\dfrac{y_{n+1}}{x_{n+1}}=\limsup\limits_{n\to\infty}\left(\dfrac{1}{2}\sqrt{\dfrac{x_n}{y_n}}+\dfrac{1}{2}\sqrt{\dfrac{y_n}{x_n}}\right)=\dfrac{1}{2}\left(\sqrt{\limsup\limits_{n\to\infty}\dfrac{x_n}{y_n}}+\sqrt{\limsup\limits_{n\to\infty}\dfrac{y_n}{x_n}}\right)=\dfrac{1}{2}\left(\sqrt{\limsup\limits_{n\to\infty}\dfrac{x_n}{y_n}}+\sqrt{\dfrac{1}{\liminf\limits_{n\to\infty}\dfrac{x_n}{y_n}}}\right)\\ \liminf\limits_{n\to\infty}\dfrac{y_{n+1}}{x_{n+1}}=\liminf\limits_{n\to\infty}\left(\dfrac{1}{2}\sqrt{\dfrac{x_n}{y_n}}+\dfrac{1}{2}\sqrt{\dfrac{y_n}{x_n}}\right)=\dfrac{1}{2}\left(\sqrt{\liminf\limits_{n\to\infty}\dfrac{x_n}{y_n}}+\sqrt{\liminf\limits_{n\to\infty}\dfrac{y_n}{x_n}}\right)=\dfrac{1}{2}\left(\sqrt{\liminf\limits_{n\to\infty}\dfrac{x_n}{y_n}}+\sqrt{\dfrac{1}{\limsup\limits_{n\to\infty}\dfrac{x_n}{y_n}}}\right)\end{cases}$$

$$\Leftrightarrow\begin{cases}L=\dfrac{1}{2}\left(\sqrt{L}+\dfrac{1}{\sqrt{l}}\right)\\ l=\dfrac{1}{2}\left(\sqrt{l}+\dfrac{1}{\sqrt{L}}\right)\end{cases}\Rightarrow\begin{cases}L-l=\dfrac{1}{2}\left[\left(\sqrt{L}-\sqrt{l}\right)+\left(\dfrac{1}{\sqrt{l}}-\dfrac{1}{\sqrt{L}}\right)\right]=\dfrac{1}{2}\left[\left(\sqrt{L}-\sqrt{l}\right)+\dfrac{\sqrt{L}-\sqrt{l}}{\sqrt{Ll}}\right]=\dfrac{1}{2}\left(\sqrt{L}-\sqrt{l}\right)\left(1+\dfrac{1}{\sqrt{Ll}}\right)\\ \dfrac{L}{l}=\dfrac{\sqrt{L}+\dfrac{1}{\sqrt{l}}}{\sqrt{l}+\dfrac{1}{\sqrt{L}}}=\sqrt{\dfrac{L}{l}}\Leftrightarrow L=l\end{cases}$$

$$\Rightarrow 2\left(\sqrt{L}+\sqrt{l}\right)=1+\frac{1}{\sqrt{Ll}}\Leftrightarrow 2\sqrt{L}=1+\frac{1}{L}\Leftrightarrow L=l=1$$

$$\Rightarrow\lim_{n\to\infty}\frac{x_n}{y_n}=1\Leftrightarrow\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n$$