$1.proof:$

$\bullet x \triangleq \sin t, t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \cos t \geq 0$

$\arcsin x = \arcsin \sin t = t$

$\arctan \frac{x}{\sqrt{1-x^2}} = \arctan \frac{\sin t}{\cos t} = \arctan \tan t = t$

$\Rightarrow \arcsin x = \arctan \frac{x}{\sqrt{1-x^2}}.$

$\bullet x \triangleq \tan t, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \frac{x}{\sqrt{1+x^2}} = \frac{\tan t}{\sqrt{1+\tan^2 t}} = \sin t.$

$\arctan x = \arctan \tan t = t.$

$\arcsin \frac{x}{\sqrt{1+x^2}} = \arcsin \sin t = t.$

$\Rightarrow \arctan x = \arcsin \frac{x}{\sqrt{1+x^2}}$

$2.proof: x, y \in [0,1]$

$x \triangleq \cos\alpha, y \triangleq \cos\beta, \alpha, \beta \in [0,\pi] \Rightarrow \sqrt{1-x^2} = \sin\alpha > 0, \sqrt{1-y^2} = \sin\beta > 0$

$then \arccos x - \arccos y = \alpha - \beta$

$\arccos\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) = \arccos(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$= \arccos\cos(a-\beta) = |a-\beta|$

$\bullet when\ x \leq y \Rightarrow \alpha \geq \beta, \alpha - \beta \geq 0$

$\arccos x - \arccos y = \alpha - \beta = |a-\beta| = \arccos\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right).$

$\bullet when\ x > y \Rightarrow \alpha < \beta, \alpha - \beta < 0$

$\arccos x - \arccos y = \alpha - \beta = -|a-\beta| = -\arccos\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right).$

$3.proof: x,y\in\mathbb{R}$

$x\triangleq\tan\alpha, y\triangleq\tan\beta, \alpha,\beta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right), xy=\tan\alpha\tan\beta$

$\Rightarrow\frac{x-y}{1+xy}=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\tan\left(\alpha-\beta\right)\in\mathbb{R}\Rightarrow\alpha-\beta\in\left(-\pi,\pi\right)-\left\{-\frac{\pi}{2},\frac{\pi}{2}\right\}$

$\Rightarrow\tan\alpha\tan\beta\neq-1$

$then\ \arctan x-\arctan y=\alpha-\beta$

$\arctan\frac{x-y}{1+xy}=\arctan\tan\left(\alpha-\beta\right)$

$\bullet when\begin{cases}xy=\tan\alpha\tan\beta>-1\\x\geq0\end{cases}$

$\Leftrightarrow\begin{cases}\tan\alpha\tan\beta+1>0\\\tan\alpha\geq0\end{cases}\Leftrightarrow\begin{cases}\tan\beta>-\cot\alpha\\\tan\alpha\geq0\end{cases}$

$\Leftrightarrow\begin{cases}\tan\beta>\tan\left(\alpha-\frac{\pi}{2}\right)\\\alpha\in\left[0,\frac{\pi}{2}\right),\alpha-\frac{\pi}{2}\in\left[-\frac{\pi}{2},0\right)\end{cases}\Leftrightarrow\begin{cases}\beta>\alpha-\frac{\pi}{2}\\\alpha\in\left[0,\frac{\pi}{2}\right)\end{cases}\Rightarrow\frac{\pi}{2}>\alpha-\beta>-\frac{\pi}{2}$

$\Rightarrow\arctan\frac{x-y}{1+xy}=\arctan\tan\left(\alpha-\beta\right)=\alpha-\beta=\arctan x-\arctan y.$

$\bullet when\begin{cases}xy=\tan\alpha\tan\beta>-1\\x<0\end{cases}$

$\Leftrightarrow\begin{cases}\tan\alpha\tan\beta+1>0\\\tan\alpha\leq0\end{cases}\Leftrightarrow\begin{cases}\tan\beta<-\cot\alpha\\\tan\alpha\leq0\end{cases}$

$\Leftrightarrow\begin{cases}\tan\beta<\tan\left(\alpha+\frac{\pi}{2}\right)\\\alpha\in\left(-\frac{\pi}{2},0\right),\alpha+\frac{\pi}{2}\in\left(0,\frac{\pi}{2}\right)\end{cases}\Leftrightarrow\begin{cases}\beta<\alpha+\frac{\pi}{2}\\\alpha\in\left(-\frac{\pi}{2},0\right)\end{cases}\Rightarrow\frac{\pi}{2}>\alpha-\beta>-\frac{\pi}{2}$

$\Rightarrow\arctan\frac{x-y}{1+xy}=\arctan\tan\left(\alpha-\beta\right)=\alpha-\beta=\arctan x-\arctan y.$

$\bullet when\begin{cases}xy=\tan\alpha\tan\beta<-1\\x>0\end{cases}$

$\Leftrightarrow\begin{cases}\tan\alpha\tan\beta+1<0\\\tan\alpha>0\end{cases}\Leftrightarrow\begin{cases}\tan\beta<-\cot\alpha\\\tan\alpha>0\end{cases}$

$\Leftrightarrow\begin{cases}\tan\beta<\tan\left(\alpha-\frac{\pi}{2}\right)\\\alpha\in\left(0,\frac{\pi}{2}\right),\alpha-\frac{\pi}{2}\in\left(-\frac{\pi}{2},0\right)\end{cases}\Leftrightarrow\begin{cases}\beta<\alpha-\frac{\pi}{2}\\\alpha\in\left(0,\frac{\pi}{2}\right)\end{cases}\Rightarrow\pi>\alpha-\beta>\frac{\pi}{2}\Rightarrow0>\alpha-\beta-\pi>-\frac{\pi}{2}$

$\Rightarrow\arctan\frac{x-y}{1+xy}=\arctan\tan\left(\alpha-\beta\right)=\arctan\tan\left(\alpha-\beta-\pi\right)$

$=\alpha-\beta-\pi=\arctan x-\arctan y-\pi.$

$\Rightarrow\arctan x-\arctan y=\pi+\arctan\frac{x-y}{1+xy}.$

$\bullet when\begin{cases}xy=\tan\alpha\tan\beta<-1\\x<0\end{cases}$

$\Leftrightarrow\begin{cases}\tan\alpha\tan\beta+1<0\\\tan\alpha<0\end{cases}\Leftrightarrow\begin{cases}\tan\beta>-\cot\alpha\\\tan\alpha<0\end{cases}$

$\Leftrightarrow\begin{cases}\tan\beta>\tan\left(\alpha+\frac{\pi}{2}\right)\\\alpha\in\left(-\frac{\pi}{2},0\right),\alpha+\frac{\pi}{2}\in\left(0,\frac{\pi}{2}\right)\end{cases}\Leftrightarrow\begin{cases}\beta>\alpha+\frac{\pi}{2}\\\alpha\in\left(-\frac{\pi}{2},0\right)\end{cases}\Rightarrow-\frac{\pi}{2}>\alpha-\beta>-\pi\Rightarrow\frac{\pi}{2}>\alpha-\beta+\pi>0$

$\Rightarrow\arctan\frac{x-y}{1+xy}=\arctan\tan\left(\alpha-\beta+\pi\right)=\alpha-\beta+\pi=\arctan x-\arctan y.$

$\Rightarrow\arctan x-\arctan y=-\pi+\arctan\frac{x-y}{1+xy}.$

$To\ sum\ up:\arctan x-\arctan y=\begin{cases}\arctan\frac{x-y}{1+xy},\ \ when\ xy<-1,\\\pi+\arctan\frac{x-y}{1+xy},\ when\ xy<-1\wedge x>0,\\\pi+\arctan\frac{x-y}{1+xy},\ when\ xy<-1\wedge x<0.\end{cases}$

$4.proof:$

$$\sum_{k=1}^{n}\arctan\frac{4k}{k^4-2k^2+2}=\sum_{k=1}^{n}\arctan\frac{(k+1)^2-(k-1)^2}{1+(k+1)^2(k-1)^2}$$

$$=\sum_{k=1}^{n}\left[\arctan(k+1)^2-\arctan(k-1)^2\right]$$

$$=\arctan(n+1)^2+\arctan n^2-\frac{\pi}{4}.$$

$5.proof:$

$$claim:\inf S=\frac{1}{3},\sup S=\frac{1}{2}.$$

$$\forall m_\varepsilon\in\mathbb{N},2m_\varepsilon<2m_\varepsilon+1\le 3m_\varepsilon-1<3m_\varepsilon.$$

$$\bullet\forall x\in S,m,n\in\mathbb{N},x=\frac{m}{n}\ge\frac{m}{3m}=\frac{1}{3}.$$

$$Goal:\forall\varepsilon>0,\exists x_\varepsilon\in S,x_\varepsilon<\frac{1}{3}+\varepsilon.$$

$$let\ x_\varepsilon=\frac{m_\varepsilon}{3m_\varepsilon-1},$$

$$thus\ \ x_\varepsilon-\frac{1}{3}=\frac{m_\varepsilon}{3m_\varepsilon-1}-\frac{1}{3}=\frac{1}{3(3m_\varepsilon-1)}$$

$$NTS:\frac{1}{3(3m_\varepsilon-1)}<\varepsilon,i.e.\,m_\varepsilon>\frac{1}{9\varepsilon}+3.$$

$by\ archimedes'\ principle:$

$$\exists m_\varepsilon\in\mathbb{N},s.t.\,m_\varepsilon>\frac{1}{9\varepsilon}+3.$$

$$\Rightarrow\forall\varepsilon>0,\exists x_\varepsilon\in S,x_\varepsilon<\frac{1}{3}+\varepsilon.$$

$$Hence,\inf S=\frac{1}{3}.$$

$$\bullet\forall x\in S,m,n\in\mathbb{N},x=\frac{m}{n}\le\frac{m}{2m}=\frac{1}{2}.$$

$$Goal:\forall\varepsilon>0,\exists x_\varepsilon\in S,x_\varepsilon>\frac{1}{2}-\varepsilon.$$

$$let\ x_\varepsilon=\frac{m_\varepsilon}{2m_\varepsilon+1},$$

$$\frac{1}{2}-\frac{m_\varepsilon}{2m_\varepsilon+1}=\frac{1}{2(2m_\varepsilon+1)}$$

$$NTS:\frac{1}{2(2m_\varepsilon+1)}<\varepsilon,i.e.m_\varepsilon>\frac{1}{4\varepsilon}-\frac{1}{2}$$

$by\ archimedes'\ principle:$

$$\exists m_\varepsilon\in\mathbb{N},s.t.\,m_\varepsilon>\frac{1}{4\varepsilon}-\frac{1}{2}.$$

$$\Rightarrow\forall\varepsilon>0,\exists x_\varepsilon\in S,x_\varepsilon>\frac{1}{2}-\varepsilon.$$

$$Hence,\sup S=\frac{1}{2}.$$

6.Proof:

Existence:

Since $\cos 0 = 1, \cos \pi = -1.$

Let $\mathrm{S} = \{x \in (0,\pi) : \cos x \le a\}, T = \{x \in (0,\pi) : \cos x > a\}, a \in (-1,1),$

$(1) \bullet NTS : S \neq \varnothing.$

$$\cos(\pi - \varepsilon) \le a \Leftrightarrow \cos \varepsilon \ge -a \Leftarrow 1 - \frac{1}{2}\varepsilon^2 \ge -a \Leftrightarrow 1 + a \ge \frac{1}{2}\varepsilon^2 \Leftrightarrow 0 < \varepsilon \le \sqrt{2(1+a)}$$

$$let\ \varepsilon = \sqrt{2(1+a)} > 0 \Rightarrow \cos\left(\pi - \sqrt{2(1+a)}\right) \le a, \left(\pi - \sqrt{2(1+a)}\right) \in (0,\pi)$$

$$\Rightarrow \left(\pi - \sqrt{2(1+a)}\right) \in S \Rightarrow S \neq \varnothing.$$

$\bullet NTS : T \neq \varnothing$

$$\cos \varepsilon > a \Leftarrow 1 - \frac{1}{2}\varepsilon^2 \ge a \Leftrightarrow 1 - a \ge \frac{1}{2}\varepsilon^2 \Leftrightarrow 0 < \varepsilon \le \sqrt{2(1-a)}$$

$$let\ \varepsilon = \sqrt{2(1-a)} > 0 \Rightarrow \cos\sqrt{2(1-a)} > a, \sqrt{2(1-a)} \in (0,\pi)$$

$$\Rightarrow \sqrt{2(1-a)} \in T \Rightarrow T \neq \varnothing.$$

$(2) S \cup T = \{x \in (0,\pi) : \cos x \le a \vee \cos x > a\} = (0,\pi)$

$(3) \forall x \in T, y \in S, \cos x > a \ge \cos y, x, y \in (0,\pi) \Rightarrow x < y.$

Hence, (T,S) *is a Dedeking cut of* $(0,\pi)$.

$\exists c \in (0,\pi), s.t\ \forall x \in T, y \in S, x \le c \le y.$

Claim : $\cos c = a.$

$$\bullet If\ \cos c < a, let\ h = \min\left\{\frac{a - \cos c}{1 + \sin c}, \frac{\pi}{4}\right\} > 0$$

$$\cos c < \cos(c - h) = \cos c \cosh + \sin c \sinh = (\cos c + 1)\cos \mathrm{h} - \cos \mathrm{h} + \sin c \sin \mathrm{h}$$

$$< (\cos c + 1) - \cos \mathrm{h} + h \sin c, (\cos \mathrm{h} > 1 - \frac{1}{2}h^2, 0 < h < \frac{\pi}{2})$$

$$< (\cos c + 1) - \left(1 - \frac{1}{2}h^2\right) + h \sin c = \cos c + \frac{1}{2}h^2 + h \sin c$$

$$< \cos c + h + h \sin c \le a$$

$c - h < c, c - h \in S \Rightarrow \exists y \in S, y < c.$

Contradiction!

$$\bullet If\ \cos c > a, let\ h = \min\left\{\frac{\cos c - a}{1 + \cos c + \sin c}, \frac{\pi}{4}\right\} > 0. \left(1 + \cos c + \sin c > 0\ for\ c \in (0,\pi)\right)$$

$$\cos c > \cos(c + h) = \cos c \cos \mathrm{h} - \sin c \sin \mathrm{h} = (\cos c + 1)\cos \mathrm{h} - \cos \mathrm{h} - \sin c \sin \mathrm{h}$$

$$> (\cos c + 1)\left(1 - \frac{1}{2}h^2\right) - \cos \mathrm{h} - h \sin c > \cos c + (\cos c + 1)\left(-\frac{1}{2}h^2\right) - h \sin c$$

$$> \cos c - (\cos c + \sin c + 1)h \ge \cos c - (\cos c + \sin c + 1) \cdot \frac{\cos c - a}{1 + \cos c + \sin c} = a.$$

$c + h > c, but\ c + h \in T \Rightarrow \exists x \in T, \cos x > c.$

Contradiction!

$Hence, \cos x = a.$

$For\ x \in [0,\pi],\ a \in [-1,1], the\ formula "\cos x = a"\ has\ a\ root.$

$Uniqueness:$

$If\ \cos x_1 = \cos x_2 = a,\ x_1, x_2 \in [0,\pi], then\ x_1 = x_2.$

$Hence:$

$For\ x \in [0,\pi],\ a \in [-1,1], the\ formula "\cos x = a"\ has\ only\ one\ root!$

$7.proof:$

$\forall x \in S, x \geq \sqrt[3]{2}.$

$Goal: \forall \varepsilon > 0, \exists x_\varepsilon \in S, s.t.\ x_\varepsilon < \sqrt[3]{2} + \varepsilon.$

$Lemma: \forall x < y, x, y \in \mathbb{R}, \exists r \in \mathbb{Q}, s.t.\ x < r < y.$

$by\ archimedes'\ principle:$

$\exists n \in \mathbb{N}, s.t.\ n(y - x) > 1 \Rightarrow ny > nx + 1.$

$\exists m \in \mathbb{Z}, s.t.\ nx + 1 \geq m > nx$

$$\Rightarrow ny > nx + 1 \geq m > nx \overset{n \in \mathbb{N}}{\Rightarrow} y > \frac{m}{n} > x$$

$$let\ r = \frac{m}{n} \in \mathbb{Q}.$$

$Hence, \forall x < y, x, y \in \mathbb{R}, \exists r \in \mathbb{Q}, s.t.\ x < r < y.$

$Obviously, for\ \sqrt[3]{2} < \sqrt[3]{2} + \varepsilon, \sqrt[3]{2}, \sqrt[3]{2} + \varepsilon \in \mathbb{R}, \exists r \in \mathbb{Q}, s.t.\ \sqrt[3]{2} < r < \sqrt[3]{2} + \varepsilon.$

$let\ x_\varepsilon = r$

$Hence, \inf S = \sqrt[3]{2}.$

3.Proof:

$Let\ S=\{x:x^5+3x\le a\}$

$\bullet NTS: S\ne\varnothing.$

Since $\left(-\left|\sqrt[5]{a}\right|\right)^5+3\left(-\left|\sqrt[5]{a}\right|\right)\le\left(-|a|\right)\le a,-\left|\sqrt[5]{a}\right|\in S, thus\ S\ne\varnothing.$

$Denote\ that\ c=\sup S$

$NTS: c^5+3c=a$

$\bullet If\ c^5+3c<a, let\ h=\min\left\{\frac{1}{2},\frac{a-c^5-3c}{1+5c+10c^2+10c^3+5c^4}\right\}>0.$

$then\ (c+h)^5+3(c+h)=c^5+3c+h^5+5h^4c+10h^3c^2+10h^2c^3+5hc^4$

$<c^5+3c+h+5hc+10hc^2+10hc^3+5hc^4=c^5+3c+\left(1+5c+10c^2+10c^3+5c^4\right)h$

$\le c^5+3c+\left(1+5c+10c^2+10c^3+5c^4\right)\cdot\frac{a-c^5-3c}{1+5c+10c^2+10c^3+5c^4}=a.$

$\Rightarrow c+h\in S, c+h>c\Rightarrow c\ne\sup S.$

$Contradiction!$

$\bullet If\ c^5+3c>a, let\ h=\min\left\{\frac{1}{2},\frac{c^5+3c-a}{1+10c^2+5c^4}\right\}>0.$

$then\ (c-h)^5+3(c-h)=c^5+3c-h^5+5h^4c-10h^3c^2+10h^2c^3-5hc^4$

$>c^5+3c-h^5-10h^3c^2-5hc^4>c^5+3c-h-10hc^2-5hc^4=c^5+3c-\left(1+10c^2+5c^4\right)h$

$\ge c^5+3c-\left(1+10c^2+5c^4\right)\cdot\frac{c^5+3c-a}{1+10c^2+5c^4}=a.$

$\Rightarrow c-h\notin S, c-h<c\Rightarrow\forall y\in S, y^5+3y\le a<(c-h)^5+3(c-h)\Rightarrow y<c-h$

$\Rightarrow c-h\ is\ an\ upper\ bound\ of\ S.$

$Contradiction!$

$\Rightarrow\exists\, c\in\mathbb{R}, s.t.\ c^5+3c=a.$

$Uniqueness:$

$trivial!$

$Hence:$

$For\ x\in\mathbb{R}, a\in\mathbb{R}, the\ formula\ "x^5+3x=a"\ has\ only\ one\ root!$

$$1.\forall \varepsilon > 0, \exists N \in \mathbb{N}, N > \frac{1}{4\varepsilon} + \frac{1}{2}, s.t. \forall n > N, n \in \mathbb{N}, \left| \frac{n}{2n-1} - \frac{1}{2} \right| = \left| \frac{1}{2(2n-1)} \right| < \varepsilon \Rightarrow \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}.$$

$\varepsilon = 0.1, N = 4; \varepsilon = 0.01, N = 26; \varepsilon = 0.001, N = 251; \varepsilon = 0.0001, N = 2501.$

$$2.(1) \forall \varepsilon > 0, \exists N \in \mathbb{N}, N > \left(\frac{5}{9\varepsilon} - \frac{1}{3} \right)^2, s.t. \forall n > N, n \in \mathbb{N}, \left| \frac{2\sqrt{n}-1}{3\sqrt{n}+1} - \frac{2}{3} \right| = \left| \frac{5}{3(3\sqrt{n}+1)} \right| < \varepsilon \Rightarrow \lim_{n\to\infty} \frac{2\sqrt{n}-1}{3\sqrt{n}+1} = \frac{2}{3}.$$

$$(2) \forall \varepsilon > 0, \exists N \in \mathbb{N}, N > \frac{1}{\varepsilon^2}, s.t. \forall n > N, n \in \mathbb{N},$$

$$\left| \frac{4n+(-1)^n\sqrt{n}}{5n-2} - \frac{4}{5} \right| = \left| \frac{(-1)^n \cdot 5\sqrt{n}+8}{5(5n-2)} \right| \le \left| \frac{5\sqrt{n}+8}{5(5n-2)} \right| \le \left| \frac{\sqrt{n}+2}{5n-2} \right| \le \left| \frac{\sqrt{n}+2\sqrt{n}}{5n-2n} \right| \le \left| \frac{3\sqrt{n}}{3n} \right| = \left| \frac{1}{\sqrt{n}} \right| < \varepsilon.$$

$$\Rightarrow \lim_{n\to\infty} \frac{4n+(-1)^n\sqrt{n}}{5n-2} = \frac{4}{5}.$$

$$(3) \forall \varepsilon > 0, \exists N \in \mathbb{N}, N > \max\left\{5, 5+\log_2\left(\frac{1}{3\varepsilon}\right)\right\}, s.t. \forall n > N, n \in \mathbb{N},$$

$$\left| \frac{2^n}{n!} \right| < \left| \frac{2^n}{1\cdot 2\cdot 3\cdot 4^{n-3}} \right| = \left| \frac{1}{3\cdot 2^{n-5}} \right| < \varepsilon \Rightarrow \lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

$$(4) \forall \varepsilon > 0, \exists N \in \mathbb{N}, N > \max\left\{2, 2+2\log_2\left(\frac{1}{\varepsilon}\right)\right\}, s.t. \forall n > N, n \in \mathbb{N},$$

$$\left| \frac{n!}{n^n} \right| = \left| \left[\frac{n}{2}\right]! \frac{n!}{\left[\frac{n}{2}\right]! n^n} \right| < \left| \left[\frac{n}{2}\right]^{\left[\frac{n}{2}\right]} \frac{n^{n-\left[\frac{n}{2}\right]}}{n^n} \right| = \left| \frac{\left[\frac{n}{2}\right]^{\left[\frac{n}{2}\right]}}{n^{\left[\frac{n}{2}\right]}} \right| < \left| \frac{\left(\frac{n}{2}\right)^{\left[\frac{n}{2}\right]}}{n^{\left[\frac{n}{2}\right]}} \right| = \left| \frac{1}{2^{\left[\frac{n}{2}\right]}} \right| < \left| \frac{1}{2^{\frac{n}{2}-1}} \right| < \varepsilon \Rightarrow \lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

$$(6) a \triangleq \frac{1}{q} - 1 > 0, \forall \varepsilon > 0, \exists N \in \mathbb{N}, N > \frac{6}{a^3\varepsilon} + 3, s.t. \forall n > N, n \in \mathbb{N},$$

$$(1+a)^n = 1 + C_n^1 a + C_n^2 a^2 + \cdots + C_n^n a^n > C_n^3 a^3 = \frac{n(n-1)(n-2)}{6} a^3$$

$$\left| n^2 q^n \right| = \left| \frac{n^2}{(1+a)^n} \right| < \left| \frac{6}{a^3} \cdot \frac{n^2}{n(n-1)(n-2)} \right| < \left| \frac{6}{a^3} \cdot \frac{1}{n-3+\frac{2}{n}} \right| < \left| \frac{6}{a^3} \cdot \frac{1}{n-3} \right| < \varepsilon \Rightarrow \lim_{n\to\infty} n^2 q^n = 0.$$

$5.proof: |x_{n+1}| \le \lambda |x_n| \le \cdots \le \lambda^n |x_1| \ (0 < \lambda < 1).$

$denote\ a = \frac{1}{\lambda} - 1 > 0.$

$$\Rightarrow |x_n| \le \cdots \le \frac{(1+a)\cdot|x_1|}{(1+a)^n}.$$

$$(1+a)^n = 1 + C_n^1 a + C_n^2 a^2 + \cdots + C_n^n a^n > C_n^1 a = na$$

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, N > \frac{(1+a)\cdot|x_1|}{\varepsilon a}, s.t. \forall n > N, n \in \mathbb{N}, |x_n| \le \frac{(1+a)\cdot|x_1|}{(1+a)^n} < \frac{(1+a)\cdot|x_1|}{na} < \varepsilon$$

$$\Rightarrow \lim_{n\to\infty} x_n = 0.$$

$$6.\lim_{n\to\infty} x_n = a \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n > N, n \in \mathbb{N}, |x_n - a| < \varepsilon.$$

(1) *proof* :

$$\lim_{n\to\infty} x_n = a \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n > N, n \in \mathbb{N}, |x_n - a| < \varepsilon$$

$$|x_n| - |a| = |x_n - a + a| - |a| \le |x_n - a| + |a| - |a| = |x_n - a| < \varepsilon$$

$$|a| - |x_n| = |a - x_n + x_n| - |x_n| \le |a - x_n| + |x_n| - |x_n| = |a - x_n| < \varepsilon$$

$$\Rightarrow \left||x_n| - |a|\right| < \varepsilon \Rightarrow \lim_{n\to\infty} |x_n| = |a|.$$

(2) *proof* :

$Suppose: a \ge 0, x_n \ge 0, n = 1, 2, \cdots.$

$\bullet if\ \ a = 0,$

$$\lim_{n\to\infty} x_n = a \Leftrightarrow \forall \varepsilon^2 > 0, \exists N \in \mathbb{N}, s.t. \forall n > N, n \in \mathbb{N}, |x_n| < \varepsilon^2$$

$\forall \varepsilon > 0, \forall n > N, n \in \mathbb{N},$

$$\left|\sqrt{x_n}\right| < \sqrt{\varepsilon^2} = \varepsilon.$$

$$\Rightarrow \lim_{n\to\infty} \sqrt{x_n} = 0.$$

$\bullet if\ \ a > 0,$

$$\lim_{n\to\infty} x_n = a \Rightarrow \exists N \in \mathbb{N}, s.t. \forall n > N, n \in \mathbb{N}, |x_n - a| < \sqrt{a}\varepsilon \left(\sqrt{a}\varepsilon > 0\right)$$

$\forall \varepsilon > 0, \forall n > N, n \in \mathbb{N},$

$$\left|\sqrt{x_n} - \sqrt{a}\right| = \left|\frac{x_n - a}{\sqrt{x_n} + \sqrt{a}}\right| < \frac{|x_n - a|}{\sqrt{a}} < \varepsilon$$

$$\Rightarrow \lim_{n\to\infty} \sqrt{x_n} = \sqrt{a}.$$

$11.proof:$

$$\lim_{n\to\infty}|x_n| = l \in [0,1) \Leftrightarrow \forall\varepsilon>0, \exists N\in\mathbb{N}, s.t. \forall n>N, n\in\mathbb{N}, \left||x_n|-l\right|<\varepsilon.$$

$$let\ \varepsilon = \frac{1-l}{2}>0, thus \forall n>N, n\in\mathbb{N}, \left||x_n|-l\right|<\frac{1-l}{2}$$

$$\forall\varepsilon>0, \exists N'\in\mathbb{N}, N'>\max\left\{\log_{\left(\frac{1+l}{2}\right)}\varepsilon, N\right\},$$

$$s.t. \forall n>N, n\in\mathbb{N}, \left|x_n^n\right| = \left||x_n|^n\right| \le \left|\left(l+\frac{1-l}{2}\right)^n\right| = \left|\left(\frac{1+l}{2}\right)^n\right| < \varepsilon.$$

$$\Rightarrow \lim_{n\to\infty} x_n^n = 0.$$

$12.proof:$

without loss of generality: $let\ a=0\left(otherwise, replace\ x_n\ by\ x_n-a\right)$

$$\lim_{n\to\infty} x_n = 0 \Leftrightarrow \forall\varepsilon>0, \exists N\in\mathbb{N}, s.t. \forall n>N, n\in\mathbb{N}, |x_n|<\frac{\varepsilon}{2}.$$

$$\forall\varepsilon>0, \exists N'\in\mathbb{N}, N'>\max\left\{2\sqrt{\frac{x_1+2x_2+\cdots+Nx_N}{\varepsilon}}, N\right\}, s.t. \forall n>N', n\in\mathbb{N}, |x_n|<\varepsilon.$$

$$\left|\frac{x_1+2x_2+\cdots+nx_n}{1+2+\cdots+n}\right| = \left|\frac{x_1+2x_2+\cdots+Nx_N+(N+1)x_{N+1}+\cdots+nx_n}{1+2+\cdots+n}\right|$$

$$=\left|\frac{x_1+2x_2+\cdots+Nx_N}{1+2+\cdots+n}+\frac{(N+1)x_{N+1}+\cdots+nx_n}{1+2+\cdots+n}\right|$$

$$<\left|\frac{x_1+2x_2+\cdots+Nx_N}{1+2+\cdots+n}+\frac{(N+1)+(N+2)+\cdots+n}{1+2+\cdots+n}\cdot\frac{\varepsilon}{2}\right|$$

$$=\left|\frac{x_1+2x_2+\cdots+Nx_N}{1+2+\cdots+n}+\frac{(N+1+n)(n-N)}{(n+1)n}\cdot\frac{\varepsilon}{2}\right|$$

$$=\left|\frac{x_1+2x_2+\cdots+Nx_N}{1+2+\cdots+n}+\frac{(n+1)n-N^2-N}{(n+1)n}\cdot\frac{\varepsilon}{2}\right|$$

$$<\left|2\cdot\frac{x_1+2x_2+\cdots+Nx_N}{n(n+1)}+\frac{\varepsilon}{2}\right| < \left|2\cdot\frac{x_1+2x_2+\cdots+Nx_N}{n^2}+\frac{\varepsilon}{2}\right| < \varepsilon$$

$$\Rightarrow \lim_{n\to\infty}\frac{x_1+2x_2+\cdots+nx_n}{1+2+\cdots+n}=0.$$

$$i.e. \lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty}\frac{x_1+2x_2+\cdots+nx_n}{1+2+\cdots+n}=a.$$

$$1.(1)\lim_{n\to\infty}\frac{3n^3+2n^2-n-2}{2n^3-3n+1}=\lim_{n\to\infty}\frac{3+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}}{2-\frac{3}{n^2}+\frac{1}{n^3}}=\frac{\lim\limits_{n\to\infty}3+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}}{\lim\limits_{n\to\infty}2-\frac{3}{n^2}+\frac{1}{n^3}}=\frac{3+\lim\limits_{n\to\infty}\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}}{2+\lim\limits_{n\to\infty}-\frac{3}{n^2}+\frac{1}{n^3}}$$

$$\forall\varepsilon>0,\exists N=\left[\frac{5}{\varepsilon}\right]+1\in\mathbb{N},s.t.\forall n>N,n\in\mathbb{N},\left|\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}\right|<\left|\frac{2}{n}+\frac{1}{n^2}+\frac{2}{n^3}\right|<\left|\frac{5}{n}\right|<\varepsilon\Rightarrow\lim_{n\to\infty}\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}=0.$$

$$\forall\varepsilon>0,\exists N=\left[\frac{4}{\varepsilon}\right]+1\in\mathbb{N},s.t.\forall n>N,n\in\mathbb{N},\left|-\frac{3}{n^2}+\frac{1}{n^3}\right|<\left|\frac{3}{n^2}+\frac{1}{n^3}\right|<\left|\frac{4}{n}\right|<\varepsilon\Rightarrow\lim_{n\to\infty}-\frac{3}{n^2}+\frac{1}{n^3}=0.$$

$$\lim_{n\to\infty}\frac{3n^3+2n^2-n-2}{2n^3-3n+1}=\frac{3}{2}.$$

$$1.(3)\lim_{n\to\infty}\frac{3^n+(-2)^n}{5^n+(-3)^n}=\lim_{n\to\infty}\frac{\left(\frac{3}{5}\right)^n+\left(-\frac{2}{5}\right)^n}{1+\left(-\frac{3}{5}\right)^n}=\frac{\lim\limits_{n\to\infty}\left(\frac{3}{5}\right)^n+\lim\limits_{n\to\infty}\left(-\frac{2}{5}\right)^n}{1+\lim\limits_{n\to\infty}\left(-\frac{3}{5}\right)^n}.$$

$$\forall\varepsilon>0,\exists N_1=\left[\left|\log_{\frac{3}{5}}\varepsilon\right|\right]+1\in\mathbb{N},s.t.\forall n>N_1,n\in\mathbb{N},\left|\left(\frac{3}{5}\right)^n\right|<\varepsilon\Rightarrow\lim_{n\to\infty}\left(\frac{3}{5}\right)^n=0.$$

$$\forall\varepsilon>0,\exists N_2=\left[\left|\log_{\frac{2}{5}}\varepsilon\right|\right]+1\in\mathbb{N},s.t.\forall n>N_2,n\in\mathbb{N},\left|\left(-\frac{2}{5}\right)^n\right|=\left|\left(\frac{2}{5}\right)^n\right|<\varepsilon\Rightarrow\lim_{n\to\infty}\left(-\frac{2}{5}\right)^n=0.$$

$$\forall\varepsilon>0,\exists N_3=\left[\left|\log_{\frac{3}{5}}\varepsilon\right|\right]+1\in\mathbb{N},s.t.\forall n>N_3,n\in\mathbb{N},\left|\left(-\frac{3}{5}\right)^n\right|=\left|\left(\frac{3}{5}\right)^n\right|<\varepsilon\Rightarrow\lim_{n\to\infty}\left(-\frac{3}{5}\right)^n=0.$$

$$\lim_{n\to\infty}\frac{3^n+(-2)^n}{5^n+(-3)^n}=0.$$

$$1.(5)\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n-1}\right)=\lim_{n\to\infty}\frac{2}{\sqrt{n+1}+\sqrt{n-1}}.$$

$$\forall\varepsilon>0,\exists N=\left[\frac{1}{\varepsilon^2}\right]+1\in\mathbb{N},s.t.\forall n>N,n\in\mathbb{N},\left|\frac{2}{\sqrt{n+1}+\sqrt{n-1}}\right|<\left|\frac{2}{2\sqrt{n}}\right|<\varepsilon\Rightarrow\lim_{n\to\infty}\frac{2}{\sqrt{n+1}+\sqrt{n-1}}=0.$$

$$\Rightarrow\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n-1}\right)=0.$$

$$1.(6)\lim_{n\to\infty}\sqrt[3]{n^2}\left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)=\lim_{n\to\infty}\sqrt[3]{n^2}\frac{n+1-n}{\sqrt[3]{(n+1)^2}+\sqrt[3]{n(n+1)}+\sqrt[3]{n^2}}=\lim_{n\to\infty}\frac{1}{\sqrt[3]{\left(1+\frac{1}{n}\right)^2}+\sqrt[3]{1+\frac{1}{n}}+1}$$

$$=\frac{1}{\lim\limits_{n\to\infty}\sqrt[3]{\left(1+\frac{1}{n}\right)^2}+\lim\limits_{n\to\infty}\sqrt[3]{1+\frac{1}{n}}+1}.$$

$$\forall\varepsilon>0,\exists N_1=\left[\frac{1}{\varepsilon}\right]+1\in\mathbb{N},s.t.\forall n>N_1,n\in\mathbb{N},\left|\sqrt[3]{\left(1+\frac{1}{n}\right)^2}-1\right|=\left|\left(1+\frac{1}{n}\right)^{\frac{2}{3}}-1\right|\le\left|\left(1+\frac{1}{n}\right)-1\right|=\left|\frac{1}{n}\right|<\varepsilon\Rightarrow\lim_{n\to\infty}\sqrt[3]{\left(1+\frac{1}{n}\right)^2}=1.$$

$$\forall\varepsilon>0,\exists N_2=\left[\frac{1}{\varepsilon}\right]+1\in\mathbb{N},s.t.\forall n>N_2,n\in\mathbb{N},\left|\sqrt[3]{1+\frac{1}{n}}-1\right|=\left|\left(1+\frac{1}{n}\right)^{\frac{1}{3}}-1\right|\le\left|\left(1+\frac{1}{n}\right)-1\right|=\left|\frac{1}{n}\right|<\varepsilon\Rightarrow\lim_{n\to\infty}\sqrt[3]{1+\frac{1}{n}}=1.$$

$$\lim_{n\to\infty}\sqrt[3]{n^2}\left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)==\frac{1}{\lim\limits_{n\to\infty}\sqrt[3]{\left(1+\frac{1}{n}\right)^2}+\lim\limits_{n\to\infty}\sqrt[3]{1+\frac{1}{n}}+1}=\frac{1}{3}.$$

5.(1)由伯努利不等式:$1>\sqrt{1-\frac{1}{n}}=\left(1-\frac{1}{n}\right)^{\frac{1}{2}}>1-\frac{1}{2n}$.

$\lim\limits_{n\to\infty}\left(1-\frac{1}{2n}\right)=1$,由两边夹法则,可知:$\lim\limits_{n\to\infty}\sqrt{1-\frac{1}{n}}=1$.

5.(2)$0\le\left|\frac{\sin n!}{\sqrt{n}}\right|\le\left|\frac{1}{\sqrt{n}}\right|$

$\lim\limits_{n\to\infty}\left|\frac{1}{\sqrt{n}}\right|=0$,由两边夹法则,可知:$\lim\limits_{n\to\infty}\frac{\sin n!}{\sqrt{n}}=0$.

5.(5)$1=\frac{n}{\sqrt{n^2}}>\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}>\frac{n}{\sqrt{n^2+n}}=\frac{1}{\sqrt{1+\frac{1}{n}}}$

$\lim\limits_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}=\frac{1}{\lim\limits_{n\to\infty}\sqrt{1+\frac{1}{n}}}=1$,由两边夹法则,可知:$\lim\limits_{n\to\infty}\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}=1$.

5.(6)$2+\frac{1}{n}=\frac{2n+1}{n}>\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{(n+1)^2}}>\frac{2n+1}{\sqrt{(n+1)^2}}=\frac{2n+1}{n+1}=2-\frac{1}{n+1}$

$\lim\limits_{n\to\infty}2+\frac{1}{n}=2,\lim\limits_{n\to\infty}2-\frac{1}{n+1}=2$,由两边夹法则,可知:$\lim\limits_{n\to\infty}\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{(n+1)^2}}=2$.

5.(7)$\frac{1}{\sqrt{\frac{n}{2}-1}}=\frac{1}{\sqrt[n]{\left(\frac{n}{2}-1\right)^{\frac{n}{2}}}}\ge\frac{1}{\sqrt[n]{\left[\frac{n}{2}\right]^{n-\left[\frac{n}{2}\right]}}}>\frac{1}{\sqrt[n]{\left[\frac{n}{2}\right]!\left[\frac{n}{2}\right]^{n-\left[\frac{n}{2}\right]}}}>\frac{1}{\sqrt[n]{n!}}>0$.

$\lim\limits_{n\to\infty}\frac{1}{\sqrt{\frac{n}{2}-1}}=0$,由两边夹法则,可知:$\lim\limits_{n\to\infty}\frac{1}{\sqrt[n]{n!}}=0$.

5.(9)$\sum\limits_{k=2}^{n}\left[\frac{1}{\sqrt[k]{n^k+1}}+\frac{1}{\sqrt[k]{n^k-1}}\right]=\sum\limits_{k=2}^{n}\frac{\sqrt[k]{n^k+1}+\sqrt[k]{n^k-1}}{\sqrt[k]{n^{2k}-1}}>2\sum\limits_{k=2}^{n}\frac{\sqrt[k]{n^k-1}}{\sqrt[k]{n^{2k}}}=2\sum\limits_{k=2}^{n}\frac{\sqrt[k]{n^k-1}}{n^2}$

$=2\sum\limits_{k=2}^{n}\frac{n-1}{n^2}=2\cdot\frac{(n-1)^2}{n^2}$

$\sum\limits_{k=2}^{n}\left[\frac{1}{\sqrt[k]{n^k+1}}+\frac{1}{\sqrt[k]{n^k-1}}\right]=\sum\limits_{k=2}^{n}\frac{\sqrt[k]{n^k+1}+\sqrt[k]{n^k-1}}{\sqrt[k]{n^{2k}-1}}<2\sum\limits_{k=2}^{n}\frac{\sqrt[k]{n^k+1}}{\sqrt[k]{n^{2k}}}=2\sum\limits_{k=2}^{n}\frac{\sqrt[k]{n^k+1}}{n^2}$

$=2\sum\limits_{k=2}^{n}\frac{n+1}{n^2}=2\cdot\frac{n^2-1}{n^2}$

$\lim\limits_{n\to\infty}2\cdot\frac{(n-1)^2}{n^2}=2,\lim\limits_{n\to\infty}2\cdot\frac{n^2-1}{n^2}=2$,由两边夹法则,可知:$\lim\limits_{n\to\infty}\sum\limits_{k=2}^{n}\left[\frac{1}{\sqrt[k]{n^k+1}}+\frac{1}{\sqrt[k]{n^k-1}}\right]=2$.

7.$\max\limits_{1\le k\le m}\{a_k\}=\sqrt[n]{\left[\max\limits_{1\le k\le m}\{a_k\}\right]^n}\le\sqrt[n]{a_1^n+a_2^n+\cdots+a_m^n}$

$\le\sqrt[n]{\left[\max\limits_{1\le k\le m}\{a_k\}\right]^n+\left[\max\limits_{1\le k\le m}\{a_k\}\right]^n+\cdots+\left[\max\limits_{1\le k\le m}\{a_k\}\right]^n}$

$=\sqrt[n]{m\left[\max\limits_{1\le k\le m}\{a_k\}\right]^n}=\max\limits_{1\le k\le m}\{a_k\}\sqrt[n]{m}$

$\lim\limits_{n\to\infty}\max\limits_{1\le k\le m}\{a_k\}\sqrt[n]{m}=\max\limits_{1\le k\le m}\{a_k\}$,由两边夹法则, 可知：

$\lim\limits_{n\to\infty}\sqrt[n]{a_1^n+a_2^n+\cdots+a_m^n}=\max\limits_{1\le k\le m}\{a_k\}.$

8.证明：

$$\left.\begin{array}{l}x_n>0, n=1,2,\cdots\\ \lim\limits_{n\to\infty}x_n=a>0\end{array}\right\}\Rightarrow\max\limits_{k\ge1}\{x_k\},\min\limits_{k\ge1}\{x_k\}>0$$

$\sqrt[n]{\max\limits_{k\ge1}\{x_k\}}\ge\sqrt[n]{x_n}\ge\sqrt[n]{\min\limits_{k\ge1}\{x_k\}}$

$\lim\limits_{n\to\infty}\sqrt[n]{\max\limits_{k\ge1}\{x_k\}}=1,\lim\limits_{n\to\infty}\sqrt[n]{\min\limits_{k\ge1}\{x_k\}}=1$,由两边夹法则, 可知:$\lim\limits_{n\to\infty}\sqrt[n]{x_n}=1.$