9.18 数分作业（7.）

7.(1) Proof:

$$\bullet\begin{cases}\forall x\in(-A),x\le\sup(-A)\\ \forall\varepsilon>0,\exists x_\varepsilon\in(-A),x_\varepsilon>\sup(-A)-\varepsilon\end{cases}$$

$$\Leftrightarrow\begin{cases}\forall -x\in A,x\le\sup(-A)\\ \forall\varepsilon>0,\exists -x_\varepsilon\in A,x_\varepsilon>\sup(-A)-\varepsilon\end{cases}$$

$$\Leftrightarrow\begin{cases}\forall x\in A,x\ge-\sup(-A)\\ \forall\varepsilon>0,\exists x_\varepsilon\in A,x_\varepsilon<-\sup(-A)+\varepsilon\end{cases}$$

By Proposition 1.2.2, $-\sup(-A)=\inf A$.

$$\bullet\begin{cases}\forall x\in(-A),x\ge\inf(-A)\\ \forall\varepsilon>0,\exists x_\varepsilon\in(-A),x_\varepsilon<\inf(-A)+\varepsilon\end{cases}$$

$$\Leftrightarrow\begin{cases}\forall -x\in A,x\ge\inf(-A)\\ \forall\varepsilon>0,\exists -x_\varepsilon\in A,x_\varepsilon<\inf(-A)+\varepsilon\end{cases}$$

$$\Leftrightarrow\begin{cases}\forall x\in A,x\le-\inf(-A)\\ \forall\varepsilon>0,\exists x_\varepsilon\in A,x_\varepsilon>-\inf(-A)+\varepsilon\end{cases}$$

By Proposition 1.2.2, $-\inf(-A)=\sup A$.

(2) Proof:

$$1^\circ\begin{cases}\forall z\in(A+B),z\le\sup(A+B)\\ \forall\varepsilon>0,\exists z_\varepsilon\in(A+B),z_\varepsilon>\sup(A+B)-\varepsilon\end{cases}$$

$\bullet$ *If* $\sup(A+B)>\sup A+\sup B$,

let $\varepsilon=\sup(A+B)-(\sup A+\sup B)>0$

then $\exists z_\varepsilon\in(A+B),z_\varepsilon>\sup A+\sup B$,

$z=x+y(x\in A,y\in B)$, *thus* $z_\varepsilon\le\sup A+\sup B$.

Contradiction!

$\bullet$ *If* $\sup(A+B)<\sup A+\sup B$,

$$\textit{let }\varepsilon=\frac{(\sup A+\sup B)-\sup(A+B)}{2}>0$$

$$\textit{then }\exists x_\varepsilon\in A,x_\varepsilon>\sup A-\frac{(\sup A+\sup B)-\sup(A+B)}{2},$$

$$\exists y_\varepsilon\in B,y_\varepsilon>\sup B-\frac{(\sup A+\sup B)-\sup(A+B)}{2}.$$

$\Rightarrow\exists\varepsilon>0,s.t.z_\varepsilon=x_\varepsilon+y_\varepsilon>\sup(A+B)$.

$z_\varepsilon\in(A+B)\Rightarrow\sup(A+B)\ge z_\varepsilon>\sup(A+B)$.

Contradiction!

Hence, $\sup(A+B)=\sup A+\sup B$.

$$2^{\circ}\begin{cases}\forall z\in(A+B),z\geq\inf(A+B)\\ \forall\varepsilon>0,\exists z_{\varepsilon}\in(A+B),z_{\varepsilon}<\inf(A+B)+\varepsilon\end{cases}$$

$\bullet$*If* $\inf(A+B)<\inf A+\inf B,$

let $\varepsilon=(\inf A+\inf B)-\inf(A+B)>0$

then $\exists z_{\varepsilon}\in(A+B),z_{\varepsilon}<\inf A+\inf B$

$z=x+y(x\in A,y\in B)$, *thus* $z\geq\inf A+\inf B.$

Contradiction!

$\bullet$*If* $\inf(A+B)>\inf A+\inf B,$

let $\varepsilon=\dfrac{\inf(A+B)-(\inf A+\inf B)}{2}>0$

then $\exists x_{\varepsilon}\in A,x_{\varepsilon}<\inf A-\dfrac{\inf(A+B)-(\inf A+\inf B)}{2},$

$\exists y_{\varepsilon}\in B,y_{\varepsilon}<\inf B-\dfrac{\inf(A+B)-(\inf A+\inf B)}{2}.$

$\Rightarrow\exists\varepsilon>0,s.t.z_{\varepsilon}=x_{\varepsilon}+y_{\varepsilon}<\inf(A+B).$

$z_{\varepsilon}\in(A+B)\Rightarrow\inf(A+B)>z_{\varepsilon}\geq\inf(A+B).$

Contradiction!

Hence, $\inf(A+B)=\inf A+\inf B.$

(3): Proof: Since $\forall x \in A, y \in B, x \geq 0, y \geq 0.$

$$1^\circ \begin{cases} \forall z \in (AB), z \leq \sup(AB) \\ \forall \varepsilon > 0, \exists z_\varepsilon \in (AB), z_\varepsilon > \sup(AB) - \varepsilon \end{cases}$$

$\bullet$ *If* $\sup(AB) > \sup A \cdot \sup B,$

let $\varepsilon = \sup(AB) - \sup A \cdot \sup B > 0$

then $\exists z_\varepsilon \in (AB), z_\varepsilon > \sup A \cdot \sup B,$

$z = x \cdot y (x \in A, y \in B),$ *thus* $z_\varepsilon \leq \sup A \cdot \sup B.$

Contradiction!

$\bullet$ *If* $\sup(AB) < \sup A \cdot \sup B,$

$$let\ \varepsilon = \frac{\sup A \cdot \sup B - \sup(AB)}{\sup A + \sup B} > 0$$

$$then\ \exists x_\varepsilon \in A, x_\varepsilon > \sup A - \frac{\sup A \cdot \sup B - \sup(AB)}{\sup A + \sup B},$$

$$\exists y_\varepsilon \in B, y_\varepsilon > \sup B - \frac{\sup A \cdot \sup B - \sup(AB)}{\sup A + \sup B}.$$

$\Rightarrow \exists \varepsilon > 0, s.t.$

$$z_\varepsilon = x_\varepsilon \cdot y_\varepsilon > \left[\sup A - \frac{\sup A \cdot \sup B - \sup(AB)}{\sup A + \sup B}\right] \cdot \left[\sup B - \frac{\sup A \cdot \sup B - \sup(AB)}{\sup A + \sup B}\right]$$

$$> \sup(AB) + \left[\frac{\sup A \cdot \sup B - \sup(AB)}{\sup A + \sup B}\right]^2 > \sup(AB)$$

$z_\varepsilon \in (AB) \Rightarrow \sup(AB) \geq z_\varepsilon > \sup(AB).$

Contradiction!

Hence, $\sup(AB) = \sup A \cdot \sup B.$

$$2^\circ \begin{cases} \forall z \in (AB), z \ge \inf(AB) \\ \forall \varepsilon > 0, \exists z_\varepsilon \in (AB), z_\varepsilon < \inf(AB) + \varepsilon \end{cases}$$

$\bullet$ *If* $\inf(AB) < \inf A \cdot \inf B$,

let $\varepsilon = \inf A \cdot \inf B - \inf(AB) > 0$

then $\exists z_\varepsilon \in (AB), z_\varepsilon < \inf A \cdot \inf B$,

$z = x \cdot y(x \in A, y \in B)$, *thus* $z_\varepsilon \ge \inf A \cdot \inf B$.

Contradiction!

$\bullet$ *If* $\inf(AB) > \inf A \cdot \inf B$,

let $\varepsilon = \dfrac{\inf(AB) - \inf A \cdot \inf B}{\inf A + \inf B} > 0$

then $\exists x_\varepsilon \in A, x_\varepsilon < \inf A + \dfrac{\inf(AB) - \inf A \cdot \inf B}{\inf A + \inf B}$,

$\exists y_\varepsilon \in B, y_\varepsilon < \inf B + \dfrac{\inf(AB) - \inf A \cdot \inf B}{\inf A + \inf B}$.

$\Rightarrow \exists \varepsilon > 0, s.t.$

$$z_\varepsilon = x_\varepsilon \cdot y_\varepsilon > \left[\inf A + \frac{\inf(AB) - \inf A \cdot \inf B}{\inf A + \inf B}\right] \cdot \left[\inf B + \frac{\inf(AB) - \inf A \cdot \inf B}{\inf A + \inf B}\right]$$

$$> \inf(AB) + \left[\frac{\inf(AB) - \inf A \cdot \inf B}{\inf A + \inf B}\right]^2 > \inf(AB)$$

$z_\varepsilon \in (AB) \Rightarrow \inf(AB) \ge z_\varepsilon > \inf(AB)$.

Contradiction!

Hence, $\inf(AB) = \inf A \cdot \inf B$.

(4): Proof by contradiction:

Assume that $\sup A > \sup B\left(A \subseteq B\right)$

$$\text{then } \begin{cases} \forall x \in A, x \le \sup A \\ \forall \varepsilon > 0, \exists x_\varepsilon \in A, x_\varepsilon > \sup A - \varepsilon \end{cases}$$

Let $\varepsilon = \sup A - \sup B > 0$

$\exists x_\varepsilon \in A \subseteq B, x_\varepsilon > \sup A - \left(\sup A - \sup B\right) = \sup B$

Contradiction!

Hence, $\sup A \le \sup B$.

Assume that $\inf A < \inf B\left(A \subseteq B\right)$

$$\text{then } \begin{cases} \forall x \in A, x \ge \inf A \\ \forall \varepsilon > 0, \exists x_\varepsilon \in A, x_\varepsilon < \inf A + \varepsilon \end{cases}$$

Let $\varepsilon = \inf B - \inf A > 0$

$\exists x_\varepsilon \in A \subseteq B, x_\varepsilon < \inf A + \left(\inf B - \inf A\right) = \inf B$

Contradiction!

Hence, $\inf A \ge \inf B$.

9.21 数分作业

3.Proof:

$Let\ S=\left\{x:x^3+ax\le b\right\},T=\left\{x:x^3+ax>b\right\}$

Since $\left(-\left|\sqrt[3]{b}\right|\right)^3+a\left(-\left|\sqrt[3]{b}\right|\right)\le\left(-\left|b\right|\right)\le b,-\left|\sqrt[3]{b}\right|\in S, thus\ S\ne\varnothing.$

Since $\left(\left|\sqrt[3]{b}\right|+1\right)^3+a\left(\left|\sqrt[3]{b}\right|+1\right)>\left(\left|\sqrt[3]{b}\right|\right)^3+a\left|\sqrt[3]{b}\right|\ge b,\left(\left|\sqrt[3]{b}\right|+1\right)\in T, thus\ T\ne\varnothing.$

$S\cup T=\left\{x:x^3+ax\in\mathbb{R}\right\}=\mathbb{R},$

$\forall x\in S,\ y\in T,\ x^3+ax\le b<y^3+ay\Rightarrow x<y.$

$Hence,\left(S,T\right) is\ a\ Dedeking\ cut\ of\ \mathbb{R}.$

$There\ exists\ only\ one\ c\in\mathbb{R},\ s.t.\ \forall x\in S,\ y\in T,\ x\le c\le y.$

$\bullet If\ c^3+ac<b, let\ h=\min\left\{\frac{1}{2},\frac{b-c^3-ac}{1+3c+3c^2+a}\right\}>0.$

$then\left(c+h\right)^3+a\left(c+h\right)=c^3+ac+3h^2c+3hc^2+h^3+ah$

$<c^3+ac+3hc+3hc^2+h+ah=c^3+ac+\left(3c+3c^2+1+a\right)h$

$\le c^3+ac+\left(3c+3c^2+1+a\right)\cdot\frac{b-c^3-ac}{1+3c+3c^2+a}=b.$

$\Rightarrow\forall x\in S,\ y\in T, x\le c<c+h<y\Rightarrow\left(c+h\right)\notin S\cup T\Rightarrow S\cup T\ne\mathbb{R}.$

$Contradiction!$

$Hence, c^3+ac\ge b.$

$\bullet If\ c^3+ac>b, let\ h=\min\left\{\frac{1}{2},\frac{c^3+ac-b}{1+3c^2+a}\right\}>0.$

$then\left(c-h\right)^3+a\left(c-h\right)=c^3+ac+3h^2c-3hc^2-h^3-ah$

$>c^3+ac-3hc^2-h-ah>c^3+ac+\left(-3c^2-1-a\right)h$

$\ge c^3+ac+\left(-3c^2-1-a\right)\cdot\frac{c^3+ac-b}{1+3c^2+a}=b.$

$\Rightarrow\forall x\in S,\ y\in T, x<c-h<c\le y\Rightarrow\left(c-h\right)\notin S\cup T\Rightarrow S\cup T\ne\mathbb{R}.$

$Contradiction!$

$Hence, c^3+ac\le b.$

$\Rightarrow There\ exists\ only\ one\ c\in\mathbb{R},\ s.t.\ c^3+ac=b.$

4.Proof:

$\forall x \in A, x \leq \sup A.$

$\forall \varepsilon > 0, \exists x_\varepsilon \in A, s.t.\ x_\varepsilon > \sup A - \varepsilon.$

$\forall y \in \left\{a^x : x \in A\right\}, y \leq \sup\left\{a^x : x \in A\right\}.$

$\forall \varepsilon > 0, \exists y_\varepsilon \in \left\{a^x : x \in A\right\}, s.t.\ y_\varepsilon > \sup\left\{a^x : x \in A\right\} - \varepsilon.$

$\bullet If\ \sup\left\{a^x : x \in A\right\} < a^{\sup A}, let\ \varepsilon = \sup A - \log_a\left(\sup\left\{a^x : x \in A\right\}\right) > 0$

$then\ x_\varepsilon \in A, a^{x_\varepsilon} > a^{\sup A - \varepsilon} = a^{\log_a\left(\sup\left\{a^x : x \in A\right\}\right)} = \sup\left\{a^x : x \in A\right\} \geq a^{x_\varepsilon}.$

$Contradiction!$

$\bullet If\ \sup\left\{a^x : x \in A\right\} > a^{\sup A}, let\ \varepsilon = \sup\left\{a^x : x \in A\right\} - a^{\sup A} > 0$

$then\ y_\varepsilon \in \left\{a^x : x \in A\right\}, \log_a y_\varepsilon \in A,\ y_\varepsilon > \sup\left\{a^x : x \in A\right\} - \varepsilon = a^{\sup A} \geq a^{\log_a y_\varepsilon} = y_\varepsilon.$

$Contradiction!$

$Hence, \sup\left\{a^x : x \in A\right\} = a^{\sup A}.$

6.(2)Proof:

Existence:

Since $\cos 0 = 1, \cos\pi = -1$.

Let $S = \{x \in (0,\pi) : \cos x \le a\}, a \in (-1,1)$,

$\bullet NTS: S \ne \varnothing$.

$$\cos(\pi-\varepsilon) \le a \Leftrightarrow \cos\varepsilon \ge -a \Leftarrow 1-\frac{1}{2}\varepsilon^2 \ge -a \Leftrightarrow 1+a \ge \frac{1}{2}\varepsilon^2 \Leftrightarrow 0 < \varepsilon \le \sqrt{2(1+a)}$$

$$let\ \varepsilon = \sqrt{2(1+a)} > 0 \Rightarrow \cos\left(\pi-\sqrt{2(1+a)}\right) \le a, \left(\pi-\sqrt{2(1+a)}\right) \in (0,\pi)$$

$$\Rightarrow \left(\pi-\sqrt{2(1+a)}\right) \in S \Rightarrow S \ne \varnothing.$$

$And\ x \in (0,\pi), thus\ there\ exists\ \inf S \in (0,\pi)$.

$\bullet NTS: \inf S \in (0,\pi)$

$Assume\ that: \inf S = 0 \Rightarrow \cos\inf S = 1$.

$$let\ \varepsilon = \min\left\{\frac{\inf S - \cos a}{1+\sin a}, \frac{\pi}{4}\right\} > 0, \cos a < \cos(a-\varepsilon) \le \inf S, a-\varepsilon \in (0,\pi)$$

$\cos(a-\varepsilon) < 1$, $a-\varepsilon$ is a lower bound of $S \Rightarrow 0$ is not the infimum of S.

$Contradiction!$

$Hence, \inf S \in (0,\pi)$

$Denote\ that\ x = \inf S \in (0,\pi)$.

$Claim: \cos x = a$.

$$\bullet If\ \cos x < a, let\ h = \min\left\{\frac{a-\cos x}{1+\sin x}, \frac{\pi}{4}\right\} > 0$$

$$\cos x < \cos(x-h) = \cos x\cosh + \sin x \sinh = (\cos x+1)\cos h - \cos h + \sin x \sin h$$

$$< (\cos x+1) - \cos h + h\sin x, (\cos h > 1-\frac{1}{2}h^2, 0 < h < \frac{\pi}{2})$$

$$< (\cos x+1) - \left(1-\frac{1}{2}h^2\right) + h\sin x = \cos x + \frac{1}{2}h^2 + h\sin x$$

$$< \cos x + h + h\sin x \le a$$

$x-h < x, x-h \in S \Rightarrow x \ne \cos\inf S$.

$Contradiction!$

$$\bullet If\ \cos x > a, let\ h = \min\left\{\frac{\cos x - a}{1+\cos x+\sin x}, \frac{\pi}{4}\right\} > 0. \left(1+\cos x+\sin x > 0\ for\ x \in (0,\pi)\right)$$

$$\cos x > \cos(x+h) = \cos x\cos h - \sin x \sin h = (\cos x+1)\cos h - \cos h - \sin x \sin h$$

$$> (\cos x+1)\left(1-\frac{1}{2}h^2\right) - \cos h - h\sin x > \cos x + (\cos x+1)\left(-\frac{1}{2}h^2\right) - h\sin x$$

$$> \cos x - (\cos x+\sin x+1)h \ge \cos x - (\cos x+\sin x+1)\cdot\frac{\cos x - a}{1+\cos x+\sin x} = a.$$

$x+h > x, but\ x+h \notin S \Rightarrow x \ne \inf S$.

$Contradiction!$

Hence, $\cos x = a$.

For $x \in [0, \pi]$, $a \in [-1, 1]$, *the formula* "$\cos x = a$" *has a root*.

Uniqueness:

If $\cos x_1 = \cos x_2 = a$, $x_1, x_2 \in [0, \pi]$, *then* $x_1 = x_2$.

Hence:

For $x \in [0, \pi]$, $a \in [-1, 1]$, *the formula* "$\cos x = a$" *has only one root*!

6.(3) Proof:

Existence:

Let S=$\left\{x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right):\tan x\le a\right\},a\in\mathbb{R},$

$\bullet NTS\left(need\ to\ show\right):S\ne\varnothing.$

$Let\ \tan\left(-\frac{\pi}{2}+\varepsilon\right)<a\left(\varepsilon\to 0^{+}\right)\Rightarrow -\cot\varepsilon<a\Rightarrow -\frac{\cos\varepsilon}{\sin\varepsilon}<a\Rightarrow\frac{\cos\varepsilon}{\sin\varepsilon}>-a$

$we\ know\ that:\cos\varepsilon>1-\frac{1}{2}\varepsilon^{2},\sin\varepsilon<\varepsilon\Rightarrow\frac{\cos\varepsilon}{\sin\varepsilon}>\frac{1-\frac{1}{2}\varepsilon^{2}}{\varepsilon}$

$need\ to\ show:\frac{1-\frac{1}{2}\varepsilon^{2}}{\varepsilon}>-a,i.e.1-\frac{1}{2}\varepsilon^{2}>-a\varepsilon\Leftrightarrow 1+a\varepsilon>\frac{1}{2}\varepsilon^{2}$

$let\ \varepsilon=\frac{1}{2\left(1+|a|\right)}>0,thus\ 1+a\varepsilon>1+a\frac{1}{2\left(1+|a|\right)}\ge 1-|a|\frac{1}{2\left(1+|a|\right)}>\frac{1}{2}>\frac{1}{2}\left[\frac{1}{2\left(1+|a|\right)}\right]^{2}=\frac{1}{2}\varepsilon^{2}.$

$\Rightarrow\tan\left(-\frac{\pi}{2}+\frac{1}{2\left(1+|a|\right)}\right)<a\Rightarrow -\frac{\pi}{2}+\frac{1}{2\left(1+|a|\right)}\in S\Rightarrow S\ne\varnothing.$

$\bullet NTS:\sup S\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$

$Assume\ that\ \sup S=\frac{\pi}{2}$

$\tan a<\tan\left(a+h\right)\le\tan\left(\sup S-\varepsilon\right),\ h=\min\left\{\frac{\pi}{4}\left|\frac{\tan\left(\sup S-\varepsilon\right)-\tan a}{1+\tan\left(\sup S-\varepsilon\right)\tan a}\right|,\frac{\pi}{4},\left|\frac{\pi}{2}-a\right|\right\}>0.$

$\tan a<\tan\left(a+h\right)\le\tan\left(\sup S-\varepsilon\right),\tan\left(a+h\right)$ is a upper bound of S$\Rightarrow\frac{\pi}{2}$ is not the supremum of S.

$Denote\ that\ x=\sup S.$

$Claim:\tan x=a.$

$Lemma:x<\tan x<\frac{4}{\pi}x\left(0<x<\frac{\pi}{4}\right),the\ proof\ is\ trivial.$

$Step1:If\ \tan x<a,$

$we\ know\ that:\tan\left(x+h\right)=\frac{\tan x+\tan h}{1-\tan x\tan h}\left(h>0\right)$

$a-\tan\left(x+h\right)=a-\frac{\tan x+\tan h}{1-\tan x\tan h}=\frac{\left(a-\tan x\right)-a\tan x\tan h-\tan h}{1-\tan x\tan h}$

$=\left(a-\tan x\right)+\frac{\left(a-\tan x\right)\tan x\tan h-a\tan x\tan h-\tan h}{1-\tan x\tan h}$

$=\left(a-\tan x\right)-\frac{\left(\tan^{2}x+1\right)\tan h}{1-\tan x\tan h}$

$$Let\ h=\min\left\{\frac{\pi}{4}\left|\frac{a-\tan x}{1+a\tan x}\right|,\frac{\pi}{4},\left|\frac{\pi}{2}-x\right|\right\}>0.$$

$$0<\tan \mathrm{h}<\frac{4}{\pi}h=\frac{4}{\pi}\min\left\{\frac{\pi}{4}\left|\frac{a-\tan x}{1+a\tan x}\right|,\frac{\pi}{4},\left|\frac{\pi}{2}-x\right|\right\}=\min\left\{\left|\frac{a-\tan x}{1+a\tan x}\right|,1,\frac{4}{\pi}\left|\frac{\pi}{2}-x\right|\right\}$$

$$\text{Since }\left|\tan x\tan \mathrm{h}\right|\le\left|\tan x\tan\left|\frac{\pi}{2}-x\right|\right|=1,\tan\left(x+h\right)\in\left(\tan x,+\infty\right)$$

$$\Rightarrow\left|\tan x\tan \mathrm{h}\right|<1,1-\tan x\tan \mathrm{h}>0.$$

$$\left(a-\tan x\right)-\frac{\left(\tan^2 x+1\right)\tan \mathrm{h}}{1-\tan x\tan \mathrm{h}}=\frac{\left(a-\tan x\right)\left(1-\tan x\tan \mathrm{h}\right)-\left(\tan^2 x+1\right)\tan \mathrm{h}}{1-\tan x\tan \mathrm{h}}$$

$$=\frac{\left(a-\tan x\right)-\left(1+a\tan x\right)\tan \mathrm{h}}{1-\tan x\tan \mathrm{h}}\ge\frac{\left(a-\tan x\right)-\left|1+a\tan x\right|\tan \mathrm{h}}{1-\tan x\tan \mathrm{h}}$$

$$\ge\frac{\left(a-\tan x\right)-\left|1+a\tan x\right|\left|\frac{a-\tan x}{1+a\tan x}\right|}{1-\tan x\tan \mathrm{h}}=0$$

$$\exists h>0,s.t.\tan x<\tan\left(x+h\right)\le a$$

$$x+h>0,x+h\in S\Rightarrow x\ne\sup S.$$

Contradiction!

$Step2: If\ \tan x > a,$

$$we\ know\ that: \tan(x-h) = \frac{\tan x - \tan \mathrm{h}}{1+\tan x \tan \mathrm{h}} (h>0)$$

$$\tan(x-h) - a = \frac{\tan x - \tan \mathrm{h}}{1+\tan x \tan \mathrm{h}} - a = \frac{(\tan x - a) - a \tan x \tan \mathrm{h} - \tan \mathrm{h}}{1+\tan x \tan \mathrm{h}}$$

$$= (\tan x - a) + \frac{-(\tan x - a)\tan x \tan \mathrm{h} - a\tan x \tan \mathrm{h} - \tan \mathrm{h}}{1+\tan x \tan \mathrm{h}}$$

$$= (\tan x - a) - \frac{(\tan^2 x + 1)\tan \mathrm{h}}{1+\tan x \tan \mathrm{h}}.$$

$$Let\ h = \min\left\{\frac{\pi}{4}\left|\frac{\tan x - a}{1 + a\tan x}\right|, \frac{\pi}{4}, \left|\frac{\pi}{2} - x\right|\right\} > 0.$$

$$0 < \tan \mathrm{h} < \frac{4}{\pi} h = \frac{4}{\pi}\min\left\{\frac{\pi}{4}\left|\frac{\tan x - a}{1 + a\tan x}\right|, \frac{\pi}{4}, \left|\frac{\pi}{2} - x\right|\right\} = \min\left\{\left|\frac{\tan x - a}{1 + a\tan x}\right|, 1, \frac{4}{\pi}\left|\frac{\pi}{2} - x\right|\right\}$$

$$\text{Since } |\tan x \tan \mathrm{h}| \le \left|\tan x \tan\left|\frac{\pi}{2} - x\right|\right| = 1, \tan(x+h) \in (\tan x, +\infty)$$

$$\Rightarrow |\tan x \tan \mathrm{h}| < 1, 1 + \tan x \tan \mathrm{h} > 0.$$

$$(\tan x - a) - \frac{(\tan^2 x + 1)\tan \mathrm{h}}{1+\tan x \tan \mathrm{h}} = \frac{(\tan x - a)(1 + \tan x \tan \mathrm{h}) - (\tan^2 x + 1)\tan \mathrm{h}}{1+\tan x \tan \mathrm{h}}$$

$$= \frac{(\tan x - a) - (1 + a\tan x)\tan \mathrm{h}}{1+\tan x \tan \mathrm{h}} \ge \frac{(\tan x - a) - |1 + a\tan x|\tan \mathrm{h}}{1+\tan x \tan \mathrm{h}}$$

$$\ge \frac{(\tan x - a) - |1 + a\tan x|\left|\frac{\tan x - a}{1 + a\tan x}\right|}{1+\tan x \tan \mathrm{h}} = 0$$

$To\ sum\ up: \exists h > 0, s.t. \tan x > \tan(x-h) \ge a$

$x - h < x, but\ x - h \notin S, \forall y \in S, \tan y \le a < \tan(x-h) \Rightarrow y < x - h$

$\Rightarrow x - h\ is\ an\ upper\ bound\ of\ S.$

Contradiction!

$Hence, \tan x = a.$

$For\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), a \in \mathbb{R}, the\ formula "\tan x = a" \ has\ a\ root.$

Uniqueness:

$If\ \tan x_1 = \tan x_2 = a, x_1, x_2 \in [0, \pi], then\ x_1 = x_2.$

Hence:

$For\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), a \in \mathbb{R}, the\ formula "\tan x = a" \ has\ only\ one\ root!$

8.Proof: $\sinh x=\dfrac{e^{x}-e^{-x}}{2},\cosh x=\dfrac{e^{x}+e^{-x}}{2}.$

$$(1)\cosh^{2}x-\sinh^{2}x=\left(\frac{e^{x}+e^{-x}}{2}\right)^{2}-\left(\frac{e^{x}-e^{-x}}{2}\right)^{2}=\frac{e^{2x}+e^{-2x}+2}{4}-\frac{e^{2x}+e^{-2x}-2}{4}=1.$$

$$(2)\sinh 2x-2\sinh x\cosh x=\frac{e^{2x}-e^{-2x}}{2}-\frac{e^{x}-e^{-x}}{2}\cdot\frac{e^{x}+e^{-x}}{2}=0\Rightarrow \sinh 2x=2\sinh x\cosh x.$$

$$(3)\cosh 2x-\sinh^{2}x-\cosh^{2}x=\frac{e^{2x}+e^{-2x}}{2}-\left(\frac{e^{x}-e^{-x}}{2}\right)^{2}-\left(\frac{e^{x}+e^{-x}}{2}\right)^{2}$$

$$=\frac{e^{2x}+e^{-2x}}{2}-\frac{e^{2x}+e^{-2x}+2}{4}-\frac{e^{2x}+e^{-2x}-2}{4}=0\Rightarrow \cosh 2x=\sinh^{2}x+\cosh^{2}x$$

$$(4)\sinh(x+y)-(\sinh x\cosh y+\cosh x\sinh y)$$

$$=\frac{e^{x+y}-e^{-(x+y)}}{2}-\left(\frac{e^{x}-e^{-x}}{2}\frac{e^{y}+e^{-y}}{2}+\frac{e^{x}+e^{-x}}{2}\frac{e^{y}-e^{-y}}{2}\right)$$

$$=\frac{e^{x+y}-e^{-(x+y)}}{2}-\left(\frac{e^{x+y}-e^{y-x}+e^{x-y}-e^{-(x+y)}}{4}+\frac{e^{x+y}+e^{y-x}-e^{x-y}-e^{-(x+y)}}{4}\right)=0.$$

$$\Rightarrow \sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y.$$

$$\sinh(x-y)-(\sinh x\cosh y-\cosh x\sinh y)$$

$$=\frac{e^{x-y}-e^{-(x-y)}}{2}-\left(\frac{e^{x}-e^{-x}}{2}\frac{e^{y}+e^{-y}}{2}-\frac{e^{x}+e^{-x}}{2}\frac{e^{y}-e^{-y}}{2}\right)$$

$$=\frac{e^{x-y}-e^{-(x-y)}}{2}-\left(\frac{e^{x+y}-e^{y-x}+e^{x-y}-e^{-(x+y)}}{4}-\frac{e^{x+y}+e^{y-x}-e^{x-y}-e^{-(x+y)}}{4}\right)=0.$$

$$\Rightarrow \sinh(x-y)=\sinh x\cosh y-\cosh x\sinh y.$$

$$\cosh nx-\sinh nx=\frac{e^{nx}+e^{-nx}}{2}-\frac{e^{nx}-e^{-nx}}{2}=e^{-nx}$$

$$\Rightarrow (\cosh x-\sinh x)^{n}=\cosh nx-\sinh nx.$$

Hence, $(\cosh x\pm\sinh x)^{n}=\cosh nx\pm\sinh nx.$

$$(5)\cosh(x+y)-(\cosh x\cosh y+\sinh x\sinh y)$$

$$=\frac{e^{x+y}+e^{-(x+y)}}{2}-\left(\frac{e^{x}+e^{-x}}{2}\frac{e^{y}+e^{-y}}{2}+\frac{e^{x}-e^{-x}}{2}\frac{e^{y}-e^{-y}}{2}\right)$$

$$=\frac{e^{x+y}+e^{-(x+y)}}{2}-\left(\frac{e^{x+y}+e^{y-x}+e^{x-y}+e^{-(x+y)}}{4}+\frac{e^{x+y}-e^{y-x}-e^{x-y}+e^{-(x+y)}}{4}\right)=0.$$

$$\Rightarrow \cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y.$$

$$\cosh(x-y)-(\cosh x\cosh y-\sinh x\sinh y)$$

$$=\frac{e^{x-y}+e^{-(x-y)}}{2}-\left(\frac{e^{x}+e^{-x}}{2}\frac{e^{y}+e^{-y}}{2}-\frac{e^{x}-e^{-x}}{2}\frac{e^{y}-e^{-y}}{2}\right)$$

$$=\frac{e^{x-y}+e^{-(x-y)}}{2}-\left(\frac{e^{x+y}+e^{y-x}+e^{x-y}+e^{-(x+y)}}{4}-\frac{e^{x+y}-e^{y-x}-e^{x-y}+e^{-(x+y)}}{4}\right)=0.$$

$$\Rightarrow \cosh(x-y)=\cosh x\cosh y-\sinh x\sinh y.$$

$$(6)(\cosh x+\sinh x)^{n}=\left(\frac{e^{x}+e^{-x}}{2}+\frac{e^{x}-e^{-x}}{2}\right)^{n}=e^{nx}$$

$$\cosh nx+\sinh nx=\frac{e^{nx}+e^{-nx}}{2}+\frac{e^{nx}-e^{-nx}}{2}=e^{nx}$$

$$\Rightarrow (\cosh x+\sinh x)^{n}=\cosh nx+\sinh nx.$$

$$(\cosh x-\sinh x)^{n}=\left(\frac{e^{x}+e^{-x}}{2}-\frac{e^{x}-e^{-x}}{2}\right)^{n}=e^{-nx}$$

Alternative proof of Definition 1.3.1 (*by Dedeking cut*):

Definition 1.3.1: $\forall a > 0, \forall n \in \mathbb{N}, n \geq 2$, *the formula* "$x^n = a$" *has only one positive root*.

Lemma: $0 < b < c$ *implies* $b^n - c^n < (b-c)nb^{n-1}$ (*trivial*)

Let $S = \{x > 0 : x^n \leq a\}, T = \{x > 0 : x^n > a\}$.

Claim: (S,T) *is a Dedeking cut of* $\mathbb{R}^+$.

(1) *let* $x = \dfrac{a}{1+a} \in (0,1), x^n < x = \dfrac{a}{1+a} < a \Rightarrow \dfrac{a}{1+a} \in S \Rightarrow S \neq \varnothing$

let $x = a+1 \in (1,+\infty), x^n > x = a+1 > a \Rightarrow a+1 \in T \Rightarrow T \neq \varnothing$

(2) $S \cup T = \{x > 0 : x^n \leq a \vee x^n > a\} = \{x > 0 : x^n \in \mathbb{R}\} = \mathbb{R}^+$.

(3) $\forall x \in S, y \in T, 0 < x^n \leq a < y^n \Rightarrow 0 < x < y$

Hence, (S,T) *is a Dedeking cut of* $\mathbb{R}^+$.

then there exists $c \in \mathbb{R}^+$, *s.t.* $\forall x \in S, y \in T, x \leq c \leq y$.

we need to show: $c^n = a$

• *If* $c^n < a$, *choose* $h \in (0,1)$, *which satisfies* $h < \dfrac{a - c^n}{n(c+1)^{n-1}}$

By lemma, $(c+h)^n - c^n < hn(c+h)^{n-1} < a - c^n$

$\Rightarrow c^n < (c+h)^n < a \Rightarrow c+h \in S$, *but* $c+h \notin T \Rightarrow \forall x \in S, y \in T, x \leq c < c+h \leq y$

$\Rightarrow \forall x \in S, y \in T, x < c + \dfrac{h}{2} < y \left(c + \dfrac{h}{2} > c > 0\right) \Rightarrow S \cup T \neq \mathbb{R}^+$.

Contradiction!

• *If* $c^n > a$, *let* $\varepsilon = \dfrac{c^n - a}{nc^{n-1}}, 0 < \varepsilon < x$,

Claim: $\forall y \in S, y < c - \varepsilon$.

Otherwise, $\exists y \in S$, *s.t.* $y \geq c - \varepsilon$,

thus $c^n - y^n < c^n - (c-\varepsilon)^n < \varepsilon n c^{n-1} = c^n - a$

$\Rightarrow y^n > a \Rightarrow y \notin S$.

Contradiction!

Hence, $\forall y \in S, y < c - \varepsilon$

$\Rightarrow \forall x \in S, y \in T, x \leq c - \varepsilon < c \leq y$

$\Rightarrow \forall x \in S, y \in T, x < c - \dfrac{\varepsilon}{2} < y \Rightarrow S \cup T \neq \mathbb{R}^+$.

Contradiction!

Hence, $c^n = a$.

By exercise 1.2#5, *we know that* c *is unique*.

To sum up, for $\forall a > 0, \forall n \in \mathbb{N}, n \geq 2$, *the formula* "$x^n = a$" *has only one positive root*!

$Alternative\ proof\ of\ Theorem\ 1.3.5\ (by\ Dedeking\ cut):$

$Theorem\ 1.3.5: \forall a>0, a\neq 1, \forall b>0, the\ formula\ "a^x=b"\ has\ only\ one\ real\ root.$

$Let\ S=\{x\in\mathbb{R}: a^x\le b\}, T=\{x\in\mathbb{R}: a^x>b\}.$

$Step 1: When\ a>1,$

$Lemma: \forall a>1, \forall n\in\mathbb{N}, n\ge 2, a<\frac{a^n+n-1}{n}(trivial)$

$Claim: (S,T)\ is\ a\ Dedeking\ cut\ of\ \mathbb{R}.$

$(1)\ According\ to\ Archimedes\ principle:$

$\bullet \exists n\in\mathbb{Z}, s.t.\ n(a+1)>b^{-1}-1$

$then\ a^n>na+n-1>b^{-1}\Rightarrow a^{-n}<b\Rightarrow -n\in S\Rightarrow S\neq\varnothing.$

$\bullet \exists m\in\mathbb{Z}, s.t.\ m(a+1)>b-1$

$then\ a^m>ma+m-1>b\Rightarrow m\in T\Rightarrow T\neq\varnothing.$

$(2)\ S\cup T=\{x\in\mathbb{R}: a^x\le b\vee a^x>b\}=\{x\in\mathbb{R}: a^x\in\mathbb{R}\}=\mathbb{R}.$

$(3)\ \forall x\in S, y\in T, 0<a^x\le b<a^y\Rightarrow x<y$

$Hence, (S,T)\ is\ a\ Dedeking\ cut\ of\ \mathbb{R}.$

$then\ there\ exists\ c\in\mathbb{R}^+, s.t. \forall x\in S, y\in T, x\le c\le y.$

$we\ need\ to\ show: a^c=b$

$note\ that\ a>1\Rightarrow a^{\frac{1}{n}}>1\Rightarrow a>n\left(a^{\frac{1}{n}}-1\right)+1.$

$\bullet If\ a^c<b, then\ a^{-c}b>1, thus\ \exists n\in\mathbb{N}, n\ge 2, s.t.\ n>\frac{a-1}{a^{-c}b-1}.$

$$n\left(a^{-c}b-1\right)>a-1>n\left(a^{\frac{1}{n}}-1\right)$$

$$\Rightarrow a^{c+\frac{1}{n}}<b\Rightarrow c+\frac{1}{n}\in S\Rightarrow \exists x\in S, x>c.$$

$Contradiction!$

$\bullet If\ a^c>b, then\ a^cb^{-1}>1, thus\ \exists n\in\mathbb{N}, n\ge 2, s.t.\ n>\frac{a-1}{a^cb^{-1}-1}.$

$$n\left(a^cb^{-1}-1\right)>a-1>n\left(a^{\frac{1}{n}}-1\right)$$

$$\Rightarrow a^{c-\frac{1}{n}}>b\Rightarrow c-\frac{1}{n}\in T\Rightarrow \exists x\in T, x<c.$$

$Contradiction!$

$Hence, a^c=b.$

$Step2: When\ 0 < a < 1,$

$Lemma: \forall a^{-1} > 1, \forall n \in \mathbb{N}, n \geq 2,\ a^{-1} < \frac{a^{-n} + n - 1}{n} (trivial)$

$Claim: (T, S)\ is\ a\ Dedeking\ cut\ of\ \mathbb{R}.$

(1) *According to Archimedes principle*:

$\bullet \exists n \in \mathbb{Z}, s.t.\ n(a^{-1} + 1) > b^{-1} - 1$

$then\ a^{-n} > na^{-1} + n - 1 > b^{-1} \Rightarrow a^n < b \Rightarrow n \in S \Rightarrow S \neq \varnothing.$

$\bullet \exists m \in \mathbb{Z}, s.t.\ m(a^{-1} + 1) > b - 1$

$then\ a^{=m} > ma^{-1} + m - 1 > b \Rightarrow -m \in T \Rightarrow T \neq \varnothing.$

$(2) S \cup T = \{x \in \mathbb{R} : a^x \leq b \vee a^x > b\} = \{x \in \mathbb{R} : a^x \in \mathbb{R}\} = \mathbb{R}.$

$(3) \forall x \in T, y \in S, 0 < a^y \leq b < a^x \Rightarrow x < y$

$Hence, (T, S)\ is\ a\ Dedeking\ cut\ of\ \mathbb{R}.$

$then\ there\ exists\ c \in \mathbb{R}, s.t. \forall x \in T, y \in S, x \leq c \leq y.$

$we\ need\ to\ show: a^c = b$

$$note\ that\ a^{-1} > 1 \Rightarrow a^{-\frac{1}{n}} > 1 \Rightarrow a^{-1} > n\left(a^{-\frac{1}{n}} - 1\right) + 1.$$

$$\bullet If\ a^c < b, then\ a^{-c}b > 1, thus\ \exists n \in \mathbb{N}, n \geq 2, s.t.\ n > \frac{a^{-1} - 1}{a^{-c}b - 1}.$$

$$n(a^{-c}b - 1) > a^{-1} - 1 > n\left(a^{-\frac{1}{n}} - 1\right)$$

$$\Rightarrow a^{c - \frac{1}{n}} < b \Rightarrow c - \frac{1}{n} \in S \Rightarrow \exists y \in S, y < c.$$

$Contradiction!$

$$\bullet If\ a^c > b, then\ a^c b^{-1} > 1, thus\ \exists n \in \mathbb{N}, n \geq 2, s.t.\ n > \frac{a^{-1} - 1}{a^c b^{-1} - 1}.$$

$$n(a^c b^{-1} - 1) > a^{-1} - 1 > n\left(a^{-\frac{1}{n}} - 1\right)$$

$$\Rightarrow a^{c + \frac{1}{n}} > b \Rightarrow c + \frac{1}{n} \in T \Rightarrow \exists x \in T, x > c.$$

$Contradiction!$

$Hence, a^c = b.$

By exercise $1.2\#5$, *we know that c is unique.*

To sum up, for $\forall a > 0, a \neq 1, \forall b > 0$, *the formula* "$a^x = b$" *has only one real root*!