$$1.(2) 考虑 y' + y\cos x = 0 \Longrightarrow y' e^{\int \cos x dx} + y\cos x e^{\int \cos x dx} = 0$$

$$\Longrightarrow y' e^{\sin x} + y\cos x e^{\sin x} = 0 \Longrightarrow y e^{\sin x} = C \Longrightarrow y = Ce^{-\sin x}$$

常数变易： $y = C(x)e^{-\sin x}$

考虑: $\dfrac{dy}{dx} + y\cos x = e^{2x}$，

其中 $\dfrac{dy}{dx} = \dfrac{d(C(x)e^{-\sin x})}{dx} = \dfrac{d(C(x))}{dx}e^{-\sin x} + C(x)\dfrac{d(e^{-\sin x})}{dx} = \dfrac{d(C(x))}{dx}e^{-\sin x} - C(x)e^{-\sin x}\cos x$

那么 $\dfrac{d(C(x))}{dx}e^{-\sin x} - C(x)e^{-\sin x}\cos x + C(x)e^{-\sin x}\cos x = e^{2x}$

$$\Longrightarrow \frac{d(C(x))}{dx} = e^{2x+\sin x} \Longrightarrow C(x) = \int e^{2x+\sin x}dx + C\,(C为常数)$$

$$\Longrightarrow y = e^{-\sin x}\int e^{2x+\sin x}dx + Ce^{-\sin x}\,(C为常数)$$

1.(4). $y y' \sin x + \frac{1}{2}y^2 \cos x = 1$. ~~考虑~~ (前)

$\Rightarrow$ ~~$\frac{1}{2}y^2\cos x$~~ $d(\frac{1}{2}y^2\sin x)/dx = 1 \Rightarrow \frac{1}{2}y^2\sin x = x + C$.

$\Rightarrow y = \sqrt{\dfrac{2x+C}{\sin x}}$，C为常数

2. $y' = a(x)\cos^2 y \Rightarrow y'\sec^2 y = a(x) \Rightarrow$ ~~$\tan y = a(x) + C$~~

$\dfrac{d(\tan y)}{dx} = a(x) \Rightarrow \tan y = a(x) + C$ 代入 $y(0) = y_0$, $\Rightarrow$

$\tan y_0 = a(0) + C \Rightarrow C = \tan y_0 - a(0). \Rightarrow \tan y = a(x) + \tan y_0 - a(0)$

$\Rightarrow y = \arctan(a(x) + \tan y_0 - a(0)) + k\pi$. $(k \in \mathbb{Z})$

4. 设 $F(x) = y(x)e^x$. $F'(x) = [y'(x) + y(x)]e^x$

则 $\lim\limits_{x\to+\infty}(y'(x) + y(x)) = 0 \Rightarrow \lim\limits_{x\to+\infty}\dfrac{F'(x)}{e^x} = 0$.

由 ~~基~~ 于 $e^x \to +\infty\ (x \to +\infty)$，那么由 L'Hospital.

$\lim\limits_{x\to+\infty}\dfrac{F(x)}{e^x} = \lim\limits_{x\to+\infty}\dfrac{F'(x)}{e^x} = 0$ 即 $\lim\limits_{x\to+\infty} y(x) = 0$，得证！

$$5.\text{记}F(x)=\int_a^x q(t)f(t)dt\text{，由条件：}F'(x)\le q(x)f(x)+p(x)q(x)F(x)$$

$$\text{构造}c(x)=\frac{F(x)-\int_a^x q(s)f(s)e^{\int_s^x p(t)q(t)dt}ds}{e^{\int_a^x p(t)q(t)dt}}=\frac{F(x)}{e^{\int_a^x p(t)q(t)dt}}-\int_a^x q(s)f(s)e^{\int_s^a p(t)q(t)dt}ds,(x\ge a)$$

$$c'(x)=\frac{F'(x)-q(x)q(x)F(x)}{e^{\int_a^x p(t)q(t)dt}}-q(x)f(x)e^{-\int_a^x q(t)q(t)dt}=\frac{F'(x)-q(x)q(x)F(x)-q(x)f(x)}{e^{\int_a^x p(t)q(t)dt}}\le 0$$

$$\Longrightarrow c(x)\le c(a)=0\Longrightarrow F(x)\le\int_a^x q(s)f(s)e^{\int_s^x p(t)q(t)dt}ds$$

$$\Longrightarrow y(x)\le f(x)+p(x)F(x)\le f(x)+p(x)\int_a^x q(s)f(s)e^{\int_s^x p(t)q(t)dt}ds\text{，得证！}$$

$$6.proof:$$

$$\left|\int_a^b f(x)dx\right|=\left|\int_a^{\frac{a+b}{2}} f(x)dx+\int_{\frac{a+b}{2}}^b f(x)dx\right|=\left|\int_a^{\frac{a+b}{2}} f(x)d(x-a)+\int_{\frac{a+b}{2}}^b f(x)d(x-b)\right|$$

$$=\left|\int_a^{\frac{a+b}{2}}(x-a)f'(x)dx+\int_{\frac{a+b}{2}}^b(x-b)f'(x)dx\right|=\frac{1}{2}\left|\int_a^{\frac{a+b}{2}} f'(x)d(x-a)^2+\int_{\frac{a+b}{2}}^b f'(x)d(x-b)^2\right|$$

$$=\frac{1}{2}\left|\int_a^{\frac{a+b}{2}}(x-a)^2 f''(x)dx+\int_{\frac{a+b}{2}}^b(x-b)^2 f''(x)dx\right|\le\frac{1}{2}\left(\int_a^{\frac{a+b}{2}}(x-a)^2|f''(x)|dx+\int_{\frac{a+b}{2}}^b(x-b)^2|f''(x)|dx\right)$$

$$\le\frac{(b-a)^2}{8}\int_a^b|f''(x)|dx$$

$$8.(1)\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\exp\left\{\frac{\ln n!}{n}-\ln n\right\}=\exp\left\{\lim_{n\to\infty}\left(\frac{\left(n+\frac{1}{2}\right)\ln n-n+O(1)}{n}-\ln n\right)\right\}=\exp\left\{\lim_{n\to\infty}\left(-1+O\left(\frac{1}{n}\right)\right)\right\}=\frac{1}{e}.$$

$$8.(2)\lim_{n\to\infty}\frac{e\sqrt[n]{n!}-n}{e\ln n}=\frac{1}{e}\lim_{n\to\infty}\frac{1}{\frac{\ln n}{n}}\left(e\frac{\sqrt[n]{n!}}{n}-1\right)=\frac{1}{e}\lim_{n\to\infty}\frac{1}{\frac{\ln n}{n}}\left[e\cdot\exp\left\{\frac{\left(n+\frac{1}{2}\right)\ln n-n+O(1)}{n}-\ln n\right\}-1\right]$$

$$=\frac{1}{e}\lim_{n\to\infty}\frac{1}{\frac{\ln n}{n}}\left[\exp\left\{\frac{\ln n+O(1)}{2n}\right\}-1\right]=\frac{1}{e}\lim_{n\to\infty}\frac{1}{\frac{\ln n}{n}}\left[\frac{\ln n+O(1)}{2n}+O\left(\left(\frac{\ln n+O(1)}{2n}\right)^2\right)\right]$$

$$=\frac{1}{e}\lim_{n\to\infty}\frac{1}{\frac{\ln n}{n}}\left(\frac{\ln n}{2n}+O\left(\frac{1}{n}\right)\right)=\frac{1}{2e}$$

$$1.\begin{cases}\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1\\ x=c\end{cases}\Longrightarrow y=\pm\frac{b}{a}\sqrt{c^2-a^2}\Longrightarrow S=\int_{-\frac{b}{a}\sqrt{c^2-a^2}}^{\frac{b}{a}\sqrt{c^2-a^2}}dy\int_{\frac{a}{b}\sqrt{b^2+y^2}}^{c}dx=\int_{-\frac{b}{a}\sqrt{c^2-a^2}}^{\frac{b}{a}\sqrt{c^2-a^2}}\left(c-\frac{a}{b}\sqrt{b^2+y^2}\right)dy$$

$$=\int_{-\frac{b}{a}\sqrt{c^2-a^2}}^{\frac{b}{a}\sqrt{c^2-a^2}}d\left\{cy-\frac{a}{2b}\left[y\sqrt{b^2+y^2}+b^2\ln\left(\sqrt{b^2+y^2}+y\right)\right]\right\}$$

$$=\left\{\frac{bc}{a}\sqrt{c^2-a^2}-\frac{a}{2b}\left[\frac{b}{a}\sqrt{c^2-a^2}\sqrt{b^2+\frac{b^2(c^2-a^2)}{a^2}}+b^2\ln\left(\sqrt{b^2+\frac{b^2(c^2-a^2)}{a^2}}+\frac{b}{a}\sqrt{c^2-a^2}\right)\right]\right\}$$

$$-\left\{-\frac{bc}{a}\sqrt{c^2-a^2}-\frac{a}{2b}\left[-\frac{b}{a}\sqrt{c^2-a^2}\sqrt{b^2+\frac{b^2(c^2-a^2)}{a^2}}+b^2\ln\left(\sqrt{b^2+\frac{b^2(c^2-a^2)}{a^2}}-\frac{b}{a}\sqrt{c^2-a^2}\right)\right]\right\}$$

$$=\left\{\frac{bc}{a}\sqrt{c^2-a^2}-\frac{a}{2b}\left[\frac{b^2c}{a^2}\sqrt{c^2-a^2}+b^2\ln\left(\frac{bc}{a}+\frac{b}{a}\sqrt{c^2-a^2}\right)\right]\right\}$$

$$-\left\{-\frac{bc}{a}\sqrt{c^2-a^2}-\frac{a}{2b}\left[-\frac{b^2c}{a^2}\sqrt{c^2-a^2}+b^2\ln\left(\frac{bc}{a}-\frac{b}{a}\sqrt{c^2-a^2}\right)\right]\right\}$$

$$=\left\{\frac{bc}{a}\sqrt{c^2-a^2}-\left[\frac{bc}{2a}\sqrt{c^2-a^2}+\frac{ab}{2}\ln\left(\frac{bc}{a}+\frac{b}{a}\sqrt{c^2-a^2}\right)\right]\right\}$$

$$-\left\{-\frac{bc}{a}\sqrt{c^2-a^2}-\left[-\frac{bc}{2a}\sqrt{c^2-a^2}+\frac{ab}{2}\ln\left(\frac{bc}{a}-\frac{b}{a}\sqrt{c^2-a^2}\right)\right]\right\}$$

$$=\frac{bc}{a}\sqrt{c^2-a^2}-\left[\frac{bc}{2a}\sqrt{c^2-a^2}+\frac{ab}{2}\ln\left(\frac{bc}{a}+\frac{b}{a}\sqrt{c^2-a^2}\right)\right]+\frac{bc}{a}\sqrt{c^2-a^2}+\left[-\frac{bc}{2a}\sqrt{c^2-a^2}+\frac{ab}{2}\ln\left(\frac{bc}{a}-\frac{b}{a}\sqrt{c^2-a^2}\right)\right]$$

$$=\frac{bc}{a}\sqrt{c^2-a^2}+\frac{ab}{2}\ln\frac{c-\sqrt{c^2-a^2}}{c+\sqrt{c^2-a^2}}$$

$$6.S=\frac{1}{2}\int_0^{2\pi}(a\theta)^2d\theta=\frac{4}{3}\pi^3a^2$$

第一个圆面积 $=\pi(2\pi a)^2=4\pi^3a^2=3S$,得证!

$$7.(3)\ (x^2+y^2)^2-2a^2(x^2-y^2)=0$$

令 $\begin{cases}x=r(\theta)\cos\theta\\ y=r(\theta)\sin\theta\end{cases},\theta\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right)\Longrightarrow r^4(\theta)-2a^2r^2(\theta)(\cos^2\theta-\sin^2\theta)=0$

$$\Longrightarrow r^2(\theta)=2a^2\cos 2\theta>0\Longrightarrow\theta\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\cup\left(\frac{3\pi}{4},\frac{5\pi}{4}\right)$$

$$S=\frac{1}{2}\int_{\theta\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\cup\left(\frac{3\pi}{4},\frac{5\pi}{4}\right)}2a^2\cos 2\theta d\theta=\frac{1}{2}\int_{\theta\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\cup\left(\frac{3\pi}{4},\frac{5\pi}{4}\right)}a^2d\sin 2\theta=2a^2$$

$$9.(2)\begin{cases}\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\\ x=c\end{cases}\Longrightarrow y=\pm\frac{b}{a}\sqrt{a^2-c^2}$$

$$\Longrightarrow V_1=\int_c^a 4\pi y^2dx=\frac{4\pi b^2}{a^2}\int_c^a\sqrt{a^2-x^2}dx=\frac{2\pi b^2}{a^2}\int_c^a d\left[x\sqrt{a^2-x^2}+a^2\arctan\left(\frac{x}{\sqrt{a^2-x^2}}\right)\right]$$

$$=\frac{2\pi b^2}{a^2}\left[\frac{\pi a^2}{2}-c\sqrt{a^2-c^2}+a^2\arctan\left(\frac{c}{\sqrt{a^2-c^2}}\right)\right]=\pi^2b^2-\frac{2\pi b^2c\sqrt{a^2-c^2}}{a^2}+2\pi b^2\arctan\left(\frac{c}{\sqrt{a^2-c^2}}\right)$$

$$V_2=\int_{-a}^c 4\pi y^2dx=-\int_c^{-a}4\pi y^2dx=-\frac{4\pi b^2}{a^2}\int_c^{-a}\sqrt{a^2-x^2}dx=-\frac{2\pi b^2}{a^2}\int_c^{-a}d\left[x\sqrt{a^2-x^2}+a^2\arctan\left(\frac{x}{\sqrt{a^2-x^2}}\right)\right]$$

$$=-\frac{2\pi b^2}{a^2}\left[-\frac{\pi a^2}{2}-c\sqrt{a^2-c^2}+a^2\arctan\left(\frac{c}{\sqrt{a^2-c^2}}\right)\right]=\pi^2b^2+\frac{2\pi b^2c\sqrt{a^2-c^2}}{a^2}-2\pi b^2\arctan\left(\frac{c}{\sqrt{a^2-c^2}}\right)$$

$$11.V=\oiiint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1}dV=\oiiint_{x^2+y^2+z^2\le 1}|J|dV,J=\begin{pmatrix}\frac{1}{a}&&\\&\frac{1}{b}&\\&&\frac{1}{c}\end{pmatrix}$$

$$=\frac{1}{abc}\oiiint_{x^2+y^2+z^2\le 1}dV=\frac{4\pi}{3abc}$$

$$15.(2)S=\int_{-r}^r\left[2\pi\left(R+\sqrt{r^2-x^2}\right)+2\pi\left(R-\sqrt{r^2-x^2}\right)\right]dx=4\pi R\int_{-r}^r dx=8\pi Rr$$

$$16.(5)r=a(1+\cos\theta)$$

$$l=\frac{1}{2}\int_0^{2\pi}r^2d\theta=\frac{1}{2}a^2\int_0^{2\pi}(1+\cos\theta)^2d\theta=\frac{1}{2}a^2\int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)d\theta$$

$$=\frac{1}{2}a^2\left(\int_0^{2\pi}d\theta+2\int_0^{2\pi}\cos\theta d\theta+\int_0^{2\pi}\cos^2\theta d\theta\right)=\frac{1}{2}a^2\left(2\pi+\int_0^{2\pi}\frac{1+\cos 2\theta}{2}d\theta\right)$$

$$=\frac{1}{2}a^2(2\pi+\pi)=\frac{3\pi}{2}a^2$$

$$16.(8)\begin{cases}x=a\cosh t=a\dfrac{e^t+e^{-t}}{2}\\ y=a\sinh t=a\dfrac{e^t-e^{-t}}{2}\\ z=bt\end{cases}$$

$$l=\int_0^c ds=\int_0^c\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}dt=\int_0^c\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}dt$$

$$=\int_0^c\sqrt{\left(a\frac{e^t-e^{-t}}{2}\right)^2+\left(a\frac{e^t+e^{-t}}{2}\right)^2+b^2}dt=\int_0^c\sqrt{\frac{a^2}{2}(e^{2t}+e^{-2t})+b^2}dt$$

$$\overset{u=e^t}{=}\int_0^c\sqrt{\frac{a^2}{2}(u^2+u^{-2})+b^2}d\ln u=\int_1^{e^c}\frac{1}{u}\sqrt{\frac{a^2}{2}(u^2+u^{-2})+b^2}du$$

算不出来的~

$$1.m=\rho\int_{L}ds=\rho\int_{0}^{\frac{\pi}{4}}\sqrt{\left(\frac{dx}{dt}\right)^{2}+\left(\frac{dy}{dt}\right)^{2}}dt=\rho\int_{0}^{\frac{\pi}{4}}\sqrt{\left(\frac{d(a\cos t)}{dt}\right)^{2}+\left(\frac{d(a\ln(\sec t+\tan t)-a\sin t)}{dt}\right)^{2}}dt$$

$$=\rho a\int_{0}^{\frac{\pi}{4}}\sqrt{\left(\frac{d(\cos t)}{dt}\right)^{2}+\left(\frac{d(\ln(\sec t+\tan t)-\sin t)}{dt}\right)^{2}}dt=\rho a\int_{0}^{\frac{\pi}{4}}\sqrt{(-\sin t)^{2}+\left(\frac{\frac{\sin t}{\cos^{2}t}+\frac{1}{\cos^{2}t}}{\sec t+\tan t}-\cos t\right)^{2}}dt$$

$$=\rho a\int_{0}^{\frac{\pi}{4}}\sqrt{(-\sin t)^{2}+\left(\frac{1+\sin t}{(\sec t+\tan t)\cos^{2}t}-\cos t\right)^{2}}dt=\rho a\int_{0}^{\frac{\pi}{4}}\sqrt{(-\sin t)^{2}+\left(\frac{1+\sin t}{(1+\sin t)\cos t}-\cos t\right)^{2}}dt$$

$$=\rho a\int_{0}^{\frac{\pi}{4}}\sqrt{(-\sin t)^{2}+\left(\frac{1}{\cos t}-\cos t\right)^{2}}dt=\rho a\int_{0}^{\frac{\pi}{4}}\sqrt{(-\sin t)^{2}+\left(\frac{1-\cos^{2}t}{\cos t}\right)^{2}}dt$$

$$=\rho a\int_{0}^{\frac{\pi}{4}}\sqrt{(-\sin t)^{2}+\left(\frac{\sin^{2}t}{\cos t}\right)^{2}}dt=\rho a\int_{0}^{\frac{\pi}{4}}\sin t\sqrt{1+\tan^{2}t}\,dt=\rho a\int_{0}^{\frac{\pi}{4}}\tan t\,dt$$

$$=\rho a\int_{0}^{\frac{\pi}{4}}\frac{\sin t}{\cos t}dt=-\rho a\int_{0}^{\frac{\pi}{4}}\frac{1}{\cos t}d\cos t=-\rho a\int_{0}^{\frac{\pi}{4}}d\ln\cos t=\frac{\rho a\ln 2}{2}$$

$$3.ds=\sqrt{\left(\frac{dx}{d\theta}\right)^{2}+\left(\frac{dy}{d\theta}\right)^{2}}d\theta=ad\theta,M=(\pi-2\theta_{0})a\rho$$

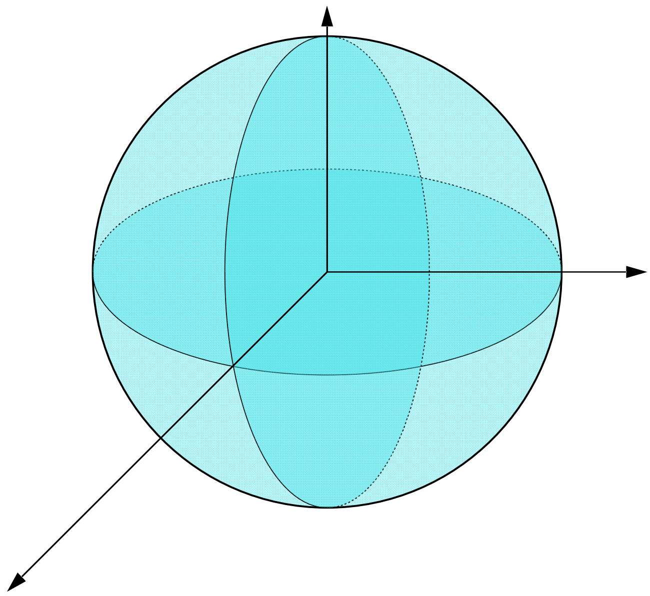
$$\bar{x}=\frac{1}{M}\int_{\theta_{0}}^{\pi-\theta_{0}}\rho a\cos\theta ds=\frac{1}{M}\int_{\theta_{0}}^{\pi-\theta_{0}}\rho a\cos\theta a d\theta=\frac{a^{2}\rho}{M}\int_{\theta_{0}}^{\pi-\theta_{0}}\cos\theta d\theta=\frac{a^{2}\rho}{(\pi-2\theta_{0})a\rho}\int_{\theta_{0}}^{\pi-\theta_{0}}\cos\theta d\theta$$

$$=\frac{a}{\pi-2\theta_{0}}\int_{\theta_{0}}^{\pi-\theta_{0}}d\sin\theta=0$$

$$\bar{y}=\frac{1}{M}\int_{\theta_{0}}^{\pi-\theta_{0}}\rho a\sin\theta ds=\frac{1}{M}\int_{\theta_{0}}^{\pi-\theta_{0}}\rho a\sin\theta a d\theta=\frac{a^{2}\rho}{M}\int_{\theta_{0}}^{\pi-\theta_{0}}\sin\theta d\theta=\frac{a^{2}\rho}{(\pi-2\theta_{0})a\rho}\int_{\theta_{0}}^{\pi-\theta_{0}}\sin\theta d\theta$$

$$=-\frac{a}{\pi-2\theta_{0}}\int_{\theta_{0}}^{\pi-\theta_{0}}d\cos\theta=\frac{2a\cos\theta_{0}}{\pi-2\theta_{0}}$$

质心的坐标为$\left(0,\frac{2a\cos\theta_{0}}{\pi-2\theta_{0}}\right)$



8.考虑上半球，即$x^2+y^2+z^2=R^2, z\geq 0$

由对称性:$\overline{x}=\overline{y}=0$,

$$M=\rho S=\rho 2\pi R^2$$

$$\overline{z}=\frac{1}{M}\int_0^R 2\pi z\sqrt{R^2-z^2}\,\rho dz=\frac{2\pi\rho}{M}\int_0^R z\sqrt{R^2-z^2}\,dz=\frac{1}{R^2}\int_0^R z\sqrt{R^2-z^2}\,dz$$

$$=\frac{1}{2R^2}\int_0^R\sqrt{R^2-z^2}\,dz^2=-\frac{1}{2R^2}\int_0^R\sqrt{R^2-z^2}\,d(R^2-z^2)=-\frac{1}{2R^2}\frac{2}{3}\int_0^R d(R^2-z^2)^{\frac{3}{2}}$$

$$=-\frac{1}{3R^2}\int_0^R d(R^2-z^2)^{\frac{3}{2}}=\frac{R}{3}$$