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$$11.(2)proof:$$

$$g(x)\text{在}f(I)\subset J\text{上一致连续}\Longrightarrow \forall\varepsilon>0,\ \exists\delta_2>0,s.t.$$

$$|g(x)-g(y)|<\varepsilon \qquad ,\forall x,y\in f(I)\subset J:|x-y|<\delta_2$$

$$f(x)\text{在}I\text{上一致连续}\Longrightarrow \exists\delta_1>0,s.t.$$

$$|f(x)-f(y)|<\delta_2 \qquad ,\forall x,y\in I:|x-y|<\delta_1$$

$$\forall\varepsilon>0,\text{取}\delta=\min\{\delta_1,\delta_2\}>0,\text{则}$$

$$|g\circ f(x)-g\circ f(y)|=|g(f(x))-g(f(y))|\overset{|f(x)-f(y)|<\delta_2}{<}\varepsilon$$

$$\text{因此，}g\circ f\text{在}I\text{上一致连续}$$

$$12.proof:$$

$$\text{记函数为}f,\text{周期为}T>0,\text{由于}f\text{在}[0,T]\subset(-\infty,+\infty)\text{连续},f\text{在}[0,T]\text{一致连续}.$$

$$\Longrightarrow \forall\varepsilon>0,\ \exists\delta>0,s.t.$$

$$|f(x)-f(y)|<\varepsilon,\forall x,y\in[0,T]:|x-y|<\delta$$

$$\text{不妨设}\delta<T,x>y$$

$$\forall x,y\in(-\infty,+\infty),$$

$$\text{由带余除法}:x=k_1T+l_1,y=k_2T+l_2,k_1,k_2\in\mathbb{Z},l_1,l_2\in[0,T)\subset[0,T]$$

$$\forall x,y\in(-\infty,+\infty):|x-y|<\delta<T,\text{有}k_1=k_2\text{或}k_1=k_2+1$$

$$①k_1=k_2\text{时},|x-y|=|l_1-l_2|<\delta$$

$$|f(x)-f(y)|=|f(k_1T+l_1)-f(k_2T+l_2)|=|f(l_1)-f(l_2)|<\varepsilon$$

$$②k_1=k_2+1\text{时},|x-y|=|T+l_1-l_2|=|l_1-(l_2-T)|<\delta$$

$$|f(x)-f(y)|=|f(k_1T+l_1)-f(k_2T+l_2)|=|f(l_1)-f(l_2)|=|f(l_1)-f(l_2-T)|<\varepsilon$$

$$\text{因此，}f\text{在}(-\infty,+\infty)\text{一致连续}.$$

$$13.proof:$$

$$\lim_{x\to+\infty}f(x)=A\Longrightarrow \forall\varepsilon>0,\ \exists M>a,s.t.|f(x)-f(y)|<\varepsilon,\forall x,y>M$$

$$f\text{在}[a,M+1]\subset[a,+\infty)\text{上连续}\Longrightarrow f\text{在}[a,M+1]\text{上一致连续}$$

$$\Longrightarrow \exists\delta>0,s.t.\forall x,y\in[a,M+1]:|x-y|<\delta,\text{有}|f(x)-f(y)|<\varepsilon$$

$$\text{不妨令}0<\delta<\frac{1}{2}\text{，那么}$$

$$\forall\varepsilon>0,\ \exists\delta>0,s.t.\forall x,y\in[a,+\infty):|x-y|<\delta,\text{有}|f(x)-f(y)|<\varepsilon.$$

$$\text{因此，}f(x)\text{在}[a,+\infty)\text{上一致连续}.$$

$16.proof$:

已知$\forall x_0 \in I, \exists \delta(x_0) > 0, C(x_0) > 0, s.t. \forall x, y \in I \cap B_{\delta(x_0)}(x_0)$,

$$|f(x) - f(y)| \le C(x_0)\,|x - y|^{\mu}$$

考虑$J(x) = \{y \in I : |f(x) - f(y)| \le C|x - y|^{\mu}\}$

因为$x \in J(x)$,那么一族所有的集合$J(x)$是一个I上的开覆盖;

又由于$I = [a,b]$是有界闭区间,是紧的,那么由有限覆盖原理,存在I中的有限点集$\{x_1, x_2, \cdots, x_n\}$

使得$I \subset J(x_1) \cup \cdots \cup J(x_n)$.

我们记$\delta = \min\{\delta(x_1), \delta(x_2), \cdots, \delta(x_n)\} > 0$,

$$C = \max\{C(x_1), C(x_2), \cdots, C(x_n)\} > 0$$

那么$\forall x_0 \in I, \forall x, y \in I \cap B_\delta(x_0), |f(x) - f(y)| \le C|x - y|^{\mu}$

那么$\forall x, y \in I : |x - y| < \delta, |f(x) - f(y)| \le C|x - y|^{\mu}$

(这里只需取$x_0 = \dfrac{x + y}{2}$，即可)

因此，f在I上全局赫尔德连续.

$1.proof$：

因为f在$[a,b]$上非负且不恒为$0 \Longrightarrow f$在(a,b)上存在$f(x_0) > 0$

由f连续性: $\exists \delta > 0, s.t. \forall y : |y - x_0| < \delta$,有

$$|f(y) - f(x_0)| < \frac{f(x_0)}{2}$$

$$i.e., f(y) > \frac{f(x_0)}{2}$$

不妨设$\delta < \min\{|b - x_0|, |a - x_0|\}$

那么$\displaystyle\int_a^b f(x)\,dx \ge \int_{x_0 - \delta}^{x_0 + \delta} f(x)\,dx > 2\delta \frac{f(x_0)}{2} = \delta f(x_0) > 0.$

2. proof：对于一个给定的$\varepsilon > 0$，取$\eta > 0$，使得$(b-a)\eta < \varepsilon$.

由于f在闭区间$[a,b]$上连续 $\Rightarrow$ f在$[a,b]$上一致连续

$\Rightarrow \exists \delta > 0$，使得$|f(x) - f(t)| < \eta$，对于任意$x, t \in [a,b]$且$|x - t| < \delta$.

对于一个$[a,b]$的划分P，满足$\Delta x_i < \delta$（对于任意i），那么

$$M_i - m_i < \eta. \quad (i = 1, 2, \dots, n)$$

因此 $U(P,f) - L(P,f) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i < \sum_{i=1}^{n}\eta\Delta x_i = \eta(b-a) < \varepsilon$.

ε任意

$\Rightarrow U(P,f) = L(P,f). \Rightarrow f \in \mathcal{R}(x)$.

$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = \int_a^b dF(x) = \sum_{i=1}^{n} 1 \cdot \Delta(F(x))_i$ 其中$\{x_0, x_1, \dots, x_n\}$是$[a,b]$一个划分

$\Delta(F(x))_i = F(x_i) - F(x_{i-1})$

$= F(b) - F(a)$.

得证!

$$3.(1)\int_0^{2\pi} x^\alpha \sin x dx = \int_0^{\pi} x^\alpha \sin x dx + \int_\pi^{2\pi} x^\alpha \sin x dx$$

$$= \xi_1 \int_0^{\pi} \sin x dx + \xi_2 \int_\pi^{2\pi} \sin x dx \ (\xi_1 \in [0,\pi], \xi_2 \in [\pi, 2\pi])$$

$$= (\xi_1 - \xi_2) \int_0^{\pi} \sin x dx \overset{\int_0^\pi \sin x dx > 0}{<} 0.$$

$$3.(3)\int_{-2}^{2} 2^x x^9 dx = \int_0^2 2^x x^9 dx + \int_{-2}^{0} 2^x x^9 dx = \int_0^2 2^x x^9 dx - \int_0^2 2^{-x} x^9 dx$$

$$= \int_0^2 (2^x - 2^{-x}) x^9 dx = \xi^9 \int_0^2 (2^x - 2^{-x}) dx \ (\xi \in (0,2)) > 0$$

$$3.(5)\int_0^{2\pi} e^{-x^2} \cos x dx = \int_0^{\pi} e^{-x^2} \cos x dx + \int_\pi^{2\pi} e^{-x^2} \cos x dx$$

$$= \int_0^{\pi} e^{-x^2} \cos x dx + \int_0^{\pi} e^{-(2\pi - x)^2} \cos x dx = \int_0^{\pi} \left(e^{-x^2} + e^{-(2\pi - x)^2}\right) \cos x dx$$

$$= \int_0^{\frac{\pi}{2}} \left(e^{-x^2} + e^{-(2\pi - x)^2}\right) \cos x dx + \int_{\frac{\pi}{2}}^{\pi} \left(e^{-x^2} + e^{-(2\pi - x)^2}\right) \cos x dx$$

$$= \left(e^{-\xi_1^2} + e^{-(2\pi - \xi_1)^2}\right) \int_0^{\frac{\pi}{2}} \cos x dx + \left(e^{-\xi_2^2} + e^{-(2\pi - \xi_2)^2}\right) \int_{\frac{\pi}{2}}^{\pi} \cos x dx \left(\xi_1 \in \left[0, \frac{\pi}{2}\right], \xi_2 \in \left[\frac{\pi}{2}, \pi\right]\right) \text{事实上,这里可以换成开区间.}$$

$$= \left[\left(e^{-\xi_1^2} + e^{-(2\pi - \xi_1)^2}\right) - \left(e^{-\xi_2^2} + e^{-(2\pi - \xi_2)^2}\right)\right] \int_0^{\frac{\pi}{2}} \cos x dx$$

$g(x) = e^{-x^2} + e^{-(2\pi - x)^2}, x \in [0,\pi]$, 易证(一点都不易证...) $g'(x) \le 0$

$$\Longrightarrow \left(e^{-\xi_1^2} + e^{-(2\pi - \xi_1)^2}\right) - \left(e^{-\xi_2^2} + e^{-(2\pi - \xi_2)^2}\right) > 0$$

$$\Longrightarrow \int_0^{2\pi} e^{-x^2} \cos x dx = \left[\left(e^{-\xi_1^2} + e^{-(2\pi - \xi_1)^2}\right) - \left(e^{-\xi_2^2} + e^{-(2\pi - \xi_2)^2}\right)\right] \int_0^{\frac{\pi}{2}} \cos x dx > 0$$

4.(1). $\int_0^{2\pi}\frac{dx}{3+\sin x^2} = \left(\int_0^{2\pi}dx\right)\frac{1}{3+\sin\xi^2}$ $(\xi\in[0,2\pi])$

$\in\left[\frac{\pi}{2},\pi\right]$.

4.(3). $\int_0^{100}\frac{e^{-x}}{x+100}dx = \frac{1}{\xi+100}\int_0^{100}e^{-x}dx$ $(\xi\in[0,100])$

$= \frac{1}{\xi+100}(-e^{-x})\Big|_0^{100}$

$= \frac{1-e^{-100}}{\xi+100}$

$\in\left[\frac{1-e^{-100}}{200},\frac{1-e^{-100}}{100}\right]$

6. $\lim_{h\to 0}\int_\alpha^\beta\frac{f(x+h)-f(x)}{h}dx = \lim_{h\to 0}\frac{1}{h}\left[\int_\alpha^\beta f(x+h)\,dx-\int_\alpha^\beta f(x)dx\right]$

$= \lim_{h\to 0}\frac{1}{h}\left[\int_{\alpha+h}^{\beta+h}f(x)dx-\int_\alpha^\beta f(x)dx\right]$

$= \lim_{h\to 0}\frac{1}{h}\left[\int_\beta^{\beta+h}f(x)dx-\int_\alpha^{\alpha+h}f(x)\,dx\right]$

$= \lim_{h\to 0}\frac{1}{h}\left[\int_0^h f(x+\beta)\,dx-\int_0^h f(x+\alpha)\,dx\right]$

$= \lim_{h\to 0}\frac{1}{h}\left[f(\xi_1+\beta)\int_0^h dx-f(\xi_2+\alpha)\int_0^h dx\right]$ $(\xi_1,\xi_2\in[0,h])$

$= \lim_{h\to 0} f(\xi_1+\beta)-f(\xi_2+\alpha) = f(\beta)-f(\alpha)$.

11.(2). $f(x)=\int_{-\tan x}^{\tan x}\frac{dt}{\sqrt{1+t^2}}$ $\left(-\frac{\pi}{2}<x<\frac{\pi}{2}\right)$

$=\int_0^{\tan x}\frac{dt}{\sqrt{1+t^2}}+\int_{-\tan x}^0\frac{dt}{\sqrt{1+t^2}}$

$=\int_0^{\tan x}\frac{dt}{\sqrt{1+t^2}}+\int_{\tan x}^0\frac{d(-t)}{\sqrt{1+t^2}}$

$=2\int_0^{\tan x}\frac{dt}{\sqrt{1+t^2}}$

$f'(x)=\frac{2}{\sqrt{1+\tan^2x}}(\tan x)'$

$=2\cos x\cdot\sec^2x=\frac{2}{\cos x}$.

$f'(x)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)}{h}$

$=\cancel{\lim_{h\to 0}\frac{2\int_0^{\tan h}\frac{dt}{\sqrt{1+t^2}}}{h}}$

$=\cancel{\lim_{h\to 0}\frac{2\tan h}{h\sqrt{1+\xi^2}}}$

11.(4) $f(x)=\int_x^{x^3}\ln^p\left(1+\sqrt{xt}\right)dt \overset{\sqrt{xt}:=u}{=} \int_x^{x^3}\ln^p(1+u)\,d\left(\frac{u^2}{x}\right)=\frac{2}{x}\int_x^{x^2}u\ln^p(1+u)\,du$

$$f'(x)=-\frac{2}{x^2}\int_x^{x^2}u\ln^p(1+u)\,du+\frac{2}{x}\left(-x\ln^p(1+x)+2x\cdot x^2\ln^p(1+x^2)\right)$$

$$=-\frac{2}{x^2}\int_x^{x^2}u\ln^p(1+u)\,du+4x^2\ln^p(1+x^2)-2\ln^p(1+x)$$

12.(1). $\lim\limits_{x\to 0.} \dfrac{\int_0^x \sin t^2 dt}{\sin^3 x} = \lim\limits_{x\to 0} \dfrac{\sin x^2}{3\sin^2 x \cos x} = \lim\limits_{x\to 0} \dfrac{x^2 - o(x^3).}{3(x - o(x^2))^2(1-o(x))} = \dfrac{1}{3}.$

12.(3). $\lim\limits_{x\to 0} \dfrac{x\left(\int_0^x e^{t^2} dt\right)^2}{\int_0^x e^{2t^2} dt} = \lim\limits_{x\to 0} \dfrac{\left(\int_0^x e^{t^2} dt\right)^2 + 2x e^{x^2} \int_0^x e^{t^2} dt}{e^{2x^2}} = \lim\limits_{x\to 0} \dfrac{\left(\int_0^x e^{t^2} dt\right)^2}{e^{2x^2}} + \lim\limits_{x\to 0} \dfrac{2x\int_0^x e^{t^2} dt}{e^{x^2}}$

$= 0 + 0 = 0$

12.(5). $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} \int_1^n \ln\left(1 + \dfrac{1}{\sqrt{t}}\right) dt \xlongequal{\text{Heine Thm}} \lim\limits_{x\to\infty} \dfrac{1}{x} \int_1^{x^2} \ln\left(1+\dfrac{1}{\sqrt{t}}\right) dt.$

$= \lim\limits_{x\to\infty} \dfrac{1}{x} \int_1^x 2t \ln\left(1+\dfrac{1}{t}\right) \cancel{dt^2} dt = 2\lim\limits_{x\to\infty} x \ln\left(1+\dfrac{1}{x}\right) = 2.$

7.4. 1. 解：我们不妨证明更强的引理：

$f \in \mathscr{R}(x)$ on $[a,b]$，$m \le f \le M$．ϕ 在 $[m,M]$ 上连续

$h(x) := \phi(f(x))$ $x \in [a,b]$，那么 $h \in \mathscr{R}(x)$ on $[a,b]$

· 对于一个给定的 $\varepsilon > 0$．ϕ 连续于 $[m,M] \Rightarrow \phi$ 在 $[m,M]$ 一致连续

$\Rightarrow \exists\, \delta > 0$ $(\delta < \varepsilon)$．s.t. $|\phi(s) - \phi(t)| < \varepsilon$ $\forall s,t \in [m,M]$ 且 $|s-t| < \delta$.

· 因为 $f \in \mathscr{R}(x)$．存在一个划分 $P = \{x_0, x_1, \dots, x_n\}$ 在 $[a,b]$ 上．s.t.

$$U(P,f) - L(P,f) < \delta^2 \quad \text{---} \quad ①$$

记 $M_i = \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$．$m_i = \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$．$U(P,f) = \sum\limits_{i=1}^{n} M_i \Delta x_i$．$L(P,f) = \sum\limits_{i=1}^{n} m_i \Delta x_i$

$\Delta x_i = x_i - x_{i-1}$ $(i = 1, \dots, n)$

$M_i^* = \sup\limits_{x \in [x_{i-1}, x_i]} h(x)$．$m_i^* = \inf\limits_{x \in [x_{i-1}, x_i]} h(x)$

记 $A = \{ i \mid M_i - m_i < \delta \}$．$B = \{ i \mid M_i - m_i \ge \delta \}$.

对于 $i \in A$．我们对 δ 的选取保证了 $M_i^* - m_i^* < \varepsilon$

对于 $i \in B$．记 $K = \sup\limits_{m \le t \le M} |\phi(t)|$．则．$M_i^* - m_i^* \le 2K$.

由①：$\delta \sum\limits_{i \in B} \Delta x_i \le \sum\limits_{i \in B} (M_i - m_i) \Delta x_i < \delta^2$

$\Rightarrow \sum\limits_{i \in B} \Delta x_i < \delta$．then, it follows that

$$U(P,h) - L(P,h) = \sum_{i \in A} (M_i^* - m_i^*) \Delta x_i + \sum_{i \in B} (M_i^* - m_i^*) \Delta x_i$$

$$\le \varepsilon(b-a) + 2K\delta < \varepsilon(b-a+2K).$$

由 ε 任意性．由达布定理：$h \in \mathscr{R}(x)$ on $[a,b]$．得证！

选取 $\phi(t) = |t| \in \mathscr{R}(x)$．原题得证！

2. (1) proof: $h_1(x)=\frac{1}{2}[f(x)+g(x)+|f(x)-g(x)|]$

$h_2(x)=\frac{1}{2}[f(x)+g(x)-|f(x)-g(x)|]$

由 ~~...~~ $f,g\in\mathscr{R}(x)$ on $[a,b]$ $\Rightarrow$ $f(x)\pm g(x)\in\mathscr{R}(x)$ on $[a,b]$

由上题 (Ex 7.4.1) 可知: $|f(x)-g(x)|\in\mathscr{R}(x)$ on $[a,b]$

$\Rightarrow$ $h_1, h_2\in\mathscr{R}(x)$ on $[a,b]$.

(2). proof: "$\Rightarrow$": 由 Ex 7.4.2(1). 显然

"$\Leftarrow$": 可以验证: $f(x)=\max\{f(x),0\}+\min\{f(x),0\}=f_+(x)+f_-(x)$.

$f_+, f_-\in\mathscr{R}(x)$ on $[a,b]$. $\Rightarrow f=f_++f_-\in\mathscr{R}(x)$ on $[a,b]$

(3). proof: 由于 $f'\in\mathscr{R}(x)$ on $[a,b]$. 则 f' 在 $[a,b]$ 有界. 记 $M=\sup\limits_{x\in[a,b]}f'(x)$, $m=\inf\limits_{x\in[a,b]}f'(x)$

取 $g(x)=|M|x-f(x)$, $h(x)=|M|x$. $g'(x)=|M|-f'(x)\geq 0$ $h'(x)=|M|\geq 0$

故 $g(x)$. $h(x)$. 均为单调递增函数 $f=h-g$. 得证!

4. proof: 证法一: 由Ex 7.4.1 中的引理: $\phi(x)=x^2$ 连续 则 $\begin{cases} f\in\mathscr{R}(x)\Rightarrow f^2\in\mathscr{R}(x) \\ g\in\mathscr{R}(x)\Rightarrow g^2\in\mathscr{R}(x)\end{cases}$

$\Rightarrow f^2+g^2\in\mathscr{R}(x)$ 由于 $\phi(x)=\sqrt{x}$ 连续 $\Rightarrow$ $\sqrt{f^2+g^2}\in\mathscr{R}(x)$. 即 $h\in\mathscr{R}(x)$. 得证!

证法二: 记 $M_1=\sup|f(x)|$. $M_2=\sup|g(x)|$. $x\in[a,b]$. $M=\max\{M_1,M_2\}$ < 这里不妨设 $f,g\geq 0$. 否则考虑 $|f|, |g|\in\mathscr{R}(x)$ (*)

• 我们断言: 对于 $[a,b]$ 上的任意划分 $P=\{x_0,x_1,\dots,x_n\}$, 有

$$\omega_i(h)\leq\sqrt{2M}\left(\sqrt{\omega_i(f)}+\sqrt{\omega_i(g)}\right).\quad i=1,2,\dots,n$$

证明: ~~$\omega_i(h)=\sup\limits_{x\in[x_{i-1},x_i]}h(x)-\inf\limits_{x\in[x_{i-1},x_i]}h(x)=\sup\limits_{x\in[x_{i-1},x_i]}\sqrt{f^2(x)+g^2(x)}$~~

对于 $x\in[x_{i-1},x_i]$. $\omega_i(h)=\sup h(x)-\inf h(x)=\sup\sqrt{f^2(x)+g^2(x)}-\inf\sqrt{f^2(x)+g^2(x)}$

$$=\sqrt{\sup[f^2(x)+g^2(x)]}-\sqrt{\inf[f^2(x)+g^2(x)]}$$

$$\leq\sqrt{\sup[f^2(x)+g^2(x)]-\inf[f^2(x)+g^2(x)]}$$

$$\leq\sqrt{[\sup f^2(x)-\inf f^2(x)]+[\sup g^2(x)-\inf g^2(x)]}$$

$$=\sqrt{[\sup|f(x)|+\inf|f(x)|][\sup|f(x)|-\inf|f(x)|]+[\sup|g(x)|+\inf|g(x)|][\sup|g(x)|-\inf|g(x)|]}$$

$$\leq\sqrt{2M}\sqrt{[\sup|f(x)|-\inf|f(x)|]+[\sup|g(x)|-\inf|g(x)|]}$$

$$\leq\sqrt{2M}\left(\sqrt{\sup|f(x)|-\inf|f(x)|}+\sqrt{\sup|g(x)|-\inf|g(x)|}\right)$$

$$=\sqrt{2M}\left(\sqrt{\omega_i(f)}+\sqrt{\omega_i(g)}\right)$$ 该等号由 (*) 保证!

得证!

~~那么~~ $\cancel{\sum_{i=1}^{n} w_i(h)\Delta x_i}$

· 我们断言，对 $[a,b]$ 上任意分割 $P=\{x_0,x_1,\dots x_n\}$ 有.

$$\sum_{i=1}^{n} w_i(h)\Delta x_i \le \sqrt{2M(b-a)}\left[\left(\sum_{i=1}^{n} w_i(f)\Delta x_i\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{n} w_i(g)\Delta x_i\right)^{\frac{1}{2}}\right].$$

证明：
$$\sum_{i=1}^{n} w_i(h)\Delta x_i \le \sum_{i=1}^{n}\sqrt{2M}\left(\sqrt{w_i(f)}+\sqrt{w_i(g)}\right)\Delta x_i$$

$$=\sqrt{2M}\sum_{i=1}^{n}\sqrt{w_i(f)}\,\Delta x_i+\sqrt{2M}\sum_{i=1}^{n}\sqrt{w_i(g)}\,\Delta x_i$$

由 Cauchy 不等式：$\sqrt{\left(\sum_{i=1}^{n}\Delta x_i\right)\left(\sum_{i=1}^{n} w_i(f)\Delta x_i\right)} \ge \sum_{i=1}^{n}\sqrt{w_i(f)}\,\Delta x_i$.

即 $\sqrt{b-a}\sqrt{\sum_{i=1}^{n} w_i(f)\Delta x_i} \ge \sum_{i=1}^{n}\sqrt{w_i(f)}\,\Delta x_i$

则上式：$\sum_{i=1}^{n} w_i(h)\Delta x_i \le \sqrt{2M(b-a)}\left[\left(\sum_{i=1}^{n} w_i(f)\Delta x_i\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{n} w_i(g)\Delta x_i\right)^{\frac{1}{2}}\right]$

· 由于 $f,g\in\mathcal{R}(x)$ on $[a,b]$. $\Rightarrow$ 对于一个~~三~~给定的 $\varepsilon>0$.

可以选取划分 P_1. s.t. $\left|\sum_{i=1}^{n} w_i(f)\Delta x_i\right| < \frac{\varepsilon^2}{8M(b-a)}$

划分 P_2. s.t. $\left|\sum_{i=1}^{n} w_i(f)\Delta x_i\right| < \frac{\varepsilon^2}{8M(b-a)}$

取 $P=P_1\cup P_2$. 则在划分 P 下.

$$\sum_{i=1}^{n} w_i(h)\Delta x_i \le \sqrt{2M(b-a)}\left[\left(\sum_{i=1}^{n} w_i(f)\Delta x_i\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{n} w_i(g)\Delta x_i\right)^{\frac{1}{2}}\right)$$

$$<\sqrt{2M(b-a)}\left\{\left[\frac{\varepsilon^2}{8M(b-a)}\right]^{\frac{1}{2}}+\left[\frac{\varepsilon^2}{8M(b-a)}\right]^{\frac{1}{2}}\right\}=\varepsilon.$$

由 ε 任意性可知：$\lim_{\|P\|\to 0}\sum_{i=1}^{n} w_i(h)\Delta x_i=0$

$\Rightarrow h\in\mathcal{R}(x)$ on $[a,b]$.

7. (1). proof：选取 $\delta\in(0,1)$ ~~选取划分 $P=\{0,\delta,1\}$~~.

~~$U(P,f)=\delta\sup_{x\in[0,\delta]} f(x)+(1-\delta)\sup_{x\in[\delta,1]} f(x)$~~

· 我们断言：f 在 $[\delta,1]$ 上可积.

证明：考虑如下一列区间：$I_k=\left[\frac{1}{k+1},\frac{1}{k}\right]$. $k=1,2,\dots$

由阿基米德原理. $\exists n\in\mathbb{N}$. s.t. $n\delta>1$. 即. $\delta>\frac{1}{n}$.

故 $[\delta,1]\subset\bigcup_{k=1}^{n} I_k$.

考虑 $x\in I_k \Rightarrow f(x)=\frac{1}{x}-k$. 连续 $\Rightarrow f\in\mathcal{R}(x)$ on I_k. $k=1,2,\dots$

事实上. $f\in\mathcal{R}(x)$ on $\bigcup_{k=1}^{n} I_k$. 只需选取 $\frac{1}{1},\frac{1}{2},\dots,\frac{1}{n}$ 为分点之一即可.

$\Rightarrow f\in\mathcal{R}(x)$ on $[\delta,1]$

·我们断言. $f \in \mathcal{R}(x)$ on $[0,\delta)$. 、对于划分 $P=\{0,\delta\}$

$$U(P,f) = \delta \sup_{x\in[0,\delta)} f(x) = \delta, \quad L(P,f) = \delta \inf_{x\in[0,\delta)} f(x) = 0.$$

$$\Rightarrow \left| U(P,f) - L(P,f) \right| = \delta$$

由 $\delta \in (0,1)$ 任意性, 令 $\delta \to 0$ 则 $|U(P,f) - L(P,f)| \to 0$

$\Rightarrow f \in \mathcal{R}(x)$ on $[0,\delta)$

Hence, $f \in \mathcal{R}(x)$ on $[0,1]$. 只需选取 δ 为分点之一即可

$$\int_0^1 f(x)\,dx = \cancel{\lim_{\delta\to 0}\int_\delta^1 f(x)\,dx = \sum_{k=1}}$$

$$= \lim_{\delta\to 0} \int_0^\delta f(x)\,dx + \int_\delta^1 f(x)\,dx$$

$$= \lim_{\delta\to 0} \int_0^\delta f(x)\,dx + \int_\delta^1 f(x)\,dx$$

$$= \lim_{\delta\to 0} \int_\delta^1 f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \int_{\frac{1}{k+1}}^{\frac{1}{k}} f(x)\,dx$$

$$\cancel{= \lim_{n\to\infty} \sum_{k=1}^{n} \int_{\frac{1}{k}}^{\frac{1}{k+1}} f(x)\,dx} = \lim_{n\to\infty} \sum_{k=1}^{n} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \left(\frac{1}{x} - \left[\frac{1}{x}\right]\right) dx$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \left(\frac{1}{x} - k\right) dx.$$ (在 $\frac{1}{k+1}$ 附近取下确界)

$$= \lim_{n\to\infty} \sum_{k=1}^{n} (\ln x - kx) \Big|_{\frac{1}{k+1}}^{\frac{1}{k}}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(-\ln k + \ln(k+1) - \frac{1}{k+1}\right)$$

$$= \lim_{n\to\infty} \left(\ln(n+1) - \sum_{k=1}^{n} \frac{1}{k+1}\right).$$

$$= \lim_{n\to\infty} \left(\ln(n+1) - \sum_{k=1}^{n+1} \frac{1}{k}\right) + 1.$$

$= 1-\gamma$. (γ 为欧拉常数)

$$11.proof:$$

$$\text{由海涅定理}:\lim_{h\to 0}\int_a^b|f(x+h)-f(x)|dx=\lim_{n\to\infty}\int_a^b\left|f\left(x+\frac{b-a}{n}\right)-f(x)\right|dx$$

将区间$[a,b]n$等分,记$x_k=a+\dfrac{b-a}{n}k$,f在$[x_k,x_{k+1}]$上的上确界为M_k,下确界为m_k,$\Delta x_k=x_{k+1}-x_k$

$$\forall\varepsilon>0,$$

由于$f\in\mathscr{R}[a,b]$,故$\exists\delta>0,s.t.$对于$[a,b]$上的任意划分,只要$\max\Delta x_k<\delta$,

$$\text{就有}\sum_{k=0}^{n-1}|M_k-m_k|\Delta x_k\le\frac{\varepsilon}{2}$$

$$\text{我们选取}n,\text{使得}\frac{b-a}{n}<\delta$$

$$\text{那么},\int_a^b\left|f\left(x+\frac{b-a}{n}\right)-f(x)\right|dx=\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}\left|f\left(x+\frac{b-a}{n}\right)-f(x)\right|dx\le\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}|M_{k+1}-m_k|dx$$

$$\le\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}|\max\{M_{k+1},M_k\}-\min\{m_{k+1},m_k\}|dx=\sum_{k=0}^{n-1}|\max\{M_{k+1},M_k\}-\min\{m_{k+1},m_k\}|\Delta x_k$$

$$=\sum_{k=0}^{n-1}(\max\{M_{k+1},M_k\}-\min\{m_{k+1},m_k\})\Delta x_k\le\sum_{k=0}^{n-1}[(M_{k+1}-m_{k+1})+(M_k-m_k)]\Delta x_k$$

$$=\sum_{k=0}^{n-1}[(M_{k+1}-m_{k+1})\Delta x_{k+1}+(M_k-m_k)\Delta x_k]\le 2\sum_{k=0}^{n-1}(M_k-m_k)\Delta x_k=\varepsilon$$

$$\text{这就说明了}:\lim_{h\to 0}\int_a^b|f(x+h)-f(x)|dx=\lim_{n\to\infty}\int_a^b\left|f\left(x+\frac{b-a}{n}\right)-f(x)\right|dx=0$$

这个做法大错特错！！！

22. 设函数 f 在闭区间 $[A,B]$ 上可积. 证明 f 具有**积分的连续性**, 即

$$\lim_{h\to 0}\int_a^b |f(x+h)-f(x)|dx=0 \quad (A<a<b<B).$$

证 对任给 $\varepsilon>0$, 因 f 在 $[A,B]$ 上可积, 故存在 $[A,B]$ 上的连续函数 φ, 使

$$\int_A^B |f(x)-\varphi(x)|dx<\frac{\varepsilon}{4}.$$

由于 φ 在 $[A,B]$ 上一致连续, 故存在 $\delta>0$, 使当 $x',x''\in[A,B], |x'-x''|<\delta$ 时, 恒有

$$|\varphi(x')-\varphi(x'')|<\frac{\varepsilon}{2(b-a)}.$$

于是, 当 $|h|<\delta$ 时,

$$\begin{aligned}\int_a^b |f(x+h)-f(x)|dx &\leqslant \int_a^b |f(x+h)-\varphi(x+h)|dx\\ &\quad+\int_a^b |\varphi(x+h)-\varphi(x)|dx+\int_a^b |\varphi(x)-f(x)|dx\\ &\leqslant 2\int_A^B |f(x)-\varphi(x)|dx+\int_a^b |\varphi(x+h)-\varphi(x)|dx\\ &<2\cdot\frac{\varepsilon}{4}+\frac{\varepsilon}{2(b-a)}(b-a)=\varepsilon.\end{aligned}$$

故

$$\lim_{h\to 0}\int_a^b |f(x+h)-f(x)|dx=0.$$