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$$9.(1)\int\frac{dx}{1+\sqrt[3]{x}}\overset{t=\sqrt[3]{x}}{=}\int\frac{3t^2dt}{1+t}=3\int\frac{(1+t)^2-2(1+t)+1}{1+t}dt$$

$$=3\int\left((1+t)-2+\frac{1}{1+t}\right)d(t+1)=\frac{3}{2}(t+1)^2-6(t+1)+3\ln(t+1)+C$$

$$=\frac{3}{2}t^2-3t+3\ln(t+1)+C=\frac{3}{2}x^{\frac{2}{3}}-3x^{\frac{1}{3}}+3\ln\left(x^{\frac{1}{3}}+1\right)+C$$

$$9.(3)\int\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx=\int\frac{\left(\sqrt{x+1}-\sqrt{x-1}\right)^2}{\left(\sqrt{x+1}+\sqrt{x-1}\right)\left(\sqrt{x+1}-\sqrt{x-1}\right)}dx$$

$$=\int\frac{x+1-2\sqrt{x^2-1}+x-1}{2}dx=\int\left(x-\sqrt{x^2-1}\right)dx=\frac{x^2}{2}-\int\sqrt{x^2-1}\,dx$$

$$let\ p=x,q=\sqrt{x^2-1},then\ p^2-q^2=1,$$

$$\int\sqrt{x^2-1}\,dx=\int qdp=\frac{1}{2}\int\left(qdp+pdq+\frac{qdp-pdq}{q^2-p^2}\right)=\frac{1}{2}\int d(pq)+\frac{1}{2}\int\frac{qdp-pdq}{q^2-p^2}$$

$$\frac{d(p+q)}{p+q}=\frac{dp}{q}=\frac{dq}{p}=\frac{qdp}{q^2}=\frac{pdq}{p^2}=\frac{qdp-pdq}{q^2-p^2}$$

$$\Longrightarrow\int\frac{qdp-pdq}{q^2-p^2}=\int\frac{d(p+q)}{p+q}=\ln(p+q)$$

$$\Longrightarrow\int\sqrt{x^2-1}\,dx=\int qdp=\frac{1}{2}pq+\frac{1}{2}\ln(p+q)=\frac{1}{2}x\sqrt{x^2-1}+\frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right)+C$$

$$\Longrightarrow\int\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx=\frac{x^2}{2}-\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right)+C$$

$$9.(4)\int\frac{x^2dx}{\sqrt{x^2+x+1}}=\int\frac{(x^2+x+1-x-1)dx}{\sqrt{x^2+x+1}}=\int\sqrt{x^2+x+1}\,dx-\int\frac{\left(x+\frac{1}{2}\right)dx}{\sqrt{x^2+x+1}}-\frac{1}{2}\int\frac{dx}{\sqrt{x^2+x+1}}$$

$$=\int\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,d\left(x+\frac{1}{2}\right)-\int d\left(\sqrt{x^2+x+1}\right)-\frac{1}{2}\int\frac{d\left(x+\frac{1}{2}\right)}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}}$$

$$let\ p=\left(x+\frac{1}{2}\right),q=\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}},then\ p^2-q^2=-\frac{3}{4},then$$

$$\int qdp=\frac{1}{2}\int d(pq)+\frac{3}{8}\int\frac{qdp-pdq}{q^2-p^2}$$

$$=\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{3}{8}\ln\left(\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\right)+C$$

$$\int\frac{d\left(x+\frac{1}{2}\right)}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}}=\frac{2}{\sqrt{3}}\int\frac{d\left(x+\frac{1}{2}\right)}{\sqrt{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1}}=\int\frac{d\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)}{\sqrt{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1}}=\operatorname{arcsinh}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

$$\Longrightarrow\int\frac{x^2dx}{\sqrt{x^2+x+1}}=\int\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,d\left(x+\frac{1}{2}\right)-\int d\left(\sqrt{x^2+x+1}\right)-\frac{1}{2}\int\frac{d\left(x+\frac{1}{2}\right)}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}}$$

$$=\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}+\frac{3}{8}\ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)-\sqrt{x^2+x+1}-\operatorname{arcsinh}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

$$=\frac{2x-3}{4}\sqrt{x^2+x+1}-\frac{1}{8}\operatorname{arcsinh}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

$$9.(6)\int\frac{dx}{x^3\sqrt{x^2+1}}$$

$$let\ p=x,q=\sqrt{x^2+1},q^2-p^2=1$$

$$then\ \int\frac{dx}{x^3\sqrt{x^2+1}}=\int\frac{dp}{p^3q}=\int\frac{dq}{p^4}=\int\frac{dq}{(q^2-1)^2}=\int\frac{\left(\frac{(q+1)-(q-1)}{2}\right)^2dq}{(q-1)^2(q+1)^2}$$

$$=\frac{1}{4}\int\frac{dq}{(q-1)^2}+\frac{1}{4}\int\frac{dq}{(q+1)^2}-\frac{1}{2}\int\frac{dq}{(q-1)(q+1)}$$

$$=\frac{1}{4}\int\frac{dq}{(q-1)^2}+\frac{1}{4}\int\frac{dq}{(q+1)^2}-\frac{1}{4}\int\frac{dq}{q-1}+\frac{1}{4}\int\frac{dq}{q+1}$$

$$=-\frac{1}{4(q-1)}-\frac{1}{4(q+1)}-\frac{1}{4}\ln|q-1|+\frac{1}{4}\ln|q+1|+C$$

$$=-\frac{q}{2(q^2-1)}+\frac{1}{4}\ln\left|\frac{q+1}{q-1}\right|+C$$

$$=-\frac{\sqrt{x^2+1}}{2x^2}+\frac{1}{4}\ln\left|\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}-1}\right|+C$$

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$$1.(1)\,proof:$$

$$0\le \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\Delta x_i)^2\le \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\Delta x_i\|\Delta\|=(b-a)\lim_{\|\Delta\|\to 0}\|\Delta\|=0$$

$$\Longrightarrow \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\Delta x_i)^2=0.$$

$$(2)\,proof:$$

$$x_{i-1}\le \xi_i\le x_i$$

$$\sum_{i=1}^{n}\xi_i\Delta x_i=\sum_{i=1}^{n}\left(\xi_i-\frac{x_{i-1}+x_i}{2}\right)\Delta x_i+\sum_{i=1}^{n}\frac{x_{i-1}+x_i}{2}\Delta x_i=\sum_{i=1}^{n}\left(\xi_i-\frac{x_{i-1}+x_i}{2}\right)\Delta x_i+\frac{1}{2}\sum_{i=1}^{n}(x_i^2-x_{i-1}^2)$$

$$=\sum_{i=1}^{n}\left(\xi_i-\frac{x_{i-1}+x_i}{2}\right)\Delta x_i+\frac{1}{2}(b^2-a^2)$$

$$\sum_{i=1}^{n}\left(\xi_i-\frac{x_{i-1}+x_i}{2}\right)\Delta x_i\le \sum_{i=1}^{n}\left(x_i-\frac{x_{i-1}+x_i}{2}\right)\Delta x_i=\frac{1}{2}\sum_{i=1}^{n}(\Delta x_i)^2$$

$$\sum_{i=1}^{n}\left(\xi_i-\frac{x_{i-1}+x_i}{2}\right)\Delta x_i\ge \sum_{i=1}^{n}\left(x_{i-1}-\frac{x_{i-1}+x_i}{2}\right)\Delta x_i=-\frac{1}{2}\sum_{i=1}^{n}(\Delta x_i)^2$$

$$\Longrightarrow \frac{1}{2}(b^2-a^2)-\frac{1}{2}\sum_{i=1}^{n}(\Delta x_i)^2\le \sum_{i=1}^{n}\xi_i\Delta x_i\le \frac{1}{2}(b^2-a^2)+\frac{1}{2}\sum_{i=1}^{n}(\Delta x_i)^2$$

$$Hence,\ \lim_{\|\Delta\|\to 0}\left[\sum_{i=1}^{n}\xi_i\Delta x_i-\frac{1}{2}(b^2-a^2)\right]=\lim_{\|\Delta\|\to 0}\left[\frac{1}{2}\sum_{i=1}^{n}(\Delta x_i)^2\right]=0$$

$$\Longrightarrow \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\xi_i\Delta x_i=\frac{1}{2}(b^2-a^2).$$

$$2.(1)\,proof:$$

$$0\le \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}|\cos\xi_i-\cos\eta_i|\Delta x_i=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\left|2\sin\frac{\xi_i+\eta_i}{2}\sin\frac{\xi_i-\eta_i}{2}\right|\Delta x_i\le \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}|\xi_i-\eta_i|\Delta x_i$$

$$\le \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\|\Delta\|\Delta x_i=(b-a)\lim_{\|\Delta\|\to 0}\|\Delta\|=0$$

$$\Longrightarrow \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}|\cos\xi_i-\cos\eta_i|\Delta x_i=0$$

$$(2)\,proof:$$

$$\left|\sum_{i=1}^{n}\cos\xi_i\Delta x_i-(\sin b-\sin a)\right|=\left|\sum_{i=1}^{n}(\cos\xi_i-\cos\eta_i)\Delta x_i+\sum_{i=1}^{n}\cos\eta_i\Delta x_i-(\sin b-\sin a)\right|$$

$$=\left|\sum_{i=1}^{n}(\cos\xi_i-\cos\eta_i)\Delta x_i+\sum_{i=1}^{n}(\sin x_i-\sin x_{i-1})-(\sin b-\sin a)\right|$$

$$=\left|\sum_{i=1}^{n}(\cos\xi_i-\cos\eta_i)\Delta x_i\right|\le\sum_{i=1}^{n}|\cos\xi_i-\cos\eta_i|\Delta x_i$$

$$\Longrightarrow 0\le \lim_{\|\Delta\|\to 0}\left|\sum_{i=1}^{n}\cos\xi_i\Delta x_i-(\sin b-\sin a)\right|\le \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}|\cos\xi_i-\cos\eta_i|\Delta x_i=0$$

$$Hence,\ \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\cos\xi_i\Delta x_i=\sin b-\sin a$$

$$8.(1)\,proof:$$

$$f\in\mathscr{R}(I),\forall I\subset[0,+\infty),which\ is\ a\ finite\ subset\ of\ [0,+\infty)\Longrightarrow\int_I f(x)dx<\infty.$$

$$\lim_{x\to+\infty}f(x)=c\Longrightarrow\forall\varepsilon>0,\ \exists X>0,s.t.$$

$$|f(x)-c|<\varepsilon,\forall x>X.$$

$$\Longrightarrow \lim_{a\to+\infty}\frac{1}{a}\int_0^a f(x)dx=\lim_{a\to+\infty}\frac{1}{a}\int_0^a f(x)dx=\lim_{a\to+\infty}\frac{1}{a}\int_0^X f(x)dx+\lim_{a\to+\infty}\frac{1}{a}\int_X^a f(x)dx$$

$$=\lim_{a\to+\infty}\frac{1}{a}\int_0^X f(x)dx+\lim_{a\to+\infty}\frac{1}{a}\int_X^a f(x)dx=\lim_{a\to+\infty}\frac{1}{a}\int_X^a|f(x)-c+c|dx$$

$$\le\lim_{a\to+\infty}\frac{1}{a}\int_X^a|f(x)-c|dx+\lim_{a\to+\infty}\frac{1}{a}\int_X^a cdx<\lim_{a\to+\infty}\frac{1}{a}\int_X^a\varepsilon dx+\lim_{a\to+\infty}\frac{1}{a}\int_X^a cdx$$

$$=\varepsilon\lim_{a\to+\infty}\frac{a-X}{a}+c\lim_{a\to+\infty}\frac{a-X}{a}=c+\varepsilon$$

$$\text{since }\varepsilon\text{ is arbitary, then }\lim_{a\to+\infty}\frac{1}{a}\int_0^a f(x)dx\le c.$$

$$\text{Similarly, }\lim_{a\to+\infty}\frac{1}{a}\int_0^a f(x)dx\ge c.$$

$$\text{Hence, }\lim_{a\to+\infty}\frac{1}{a}\int_0^a f(x)dx=c.$$

$$8.(2)\, proof:$$

$$f \in \mathscr{R}[0,T] \Longrightarrow \int_0^l f(x)dx < \infty, l \in [0,T]$$

$$①if\ a = kT, k \in \mathbb{N}, then$$

$$\lim_{a\to+\infty} \frac{1}{a}\int_0^a f(x)dx = \lim_{k\to+\infty} \frac{1}{kT}\int_0^{kT} f(x)dx = \lim_{k\to+\infty} \frac{k}{kT}\int_0^{T} f(x)dx = \lim_{k\to+\infty} \frac{1}{T}\int_0^{T} f(x)dx = \frac{1}{T}\int_0^{T} f(x)dx$$

$$②if\ a = kT + l, k \in \mathbb{N}, l \in (0,T), then$$

$$\lim_{a\to+\infty} \frac{1}{a}\int_0^a f(x)dx = \lim_{k\to+\infty} \frac{1}{kT+l}\int_0^{kT+l} f(x)dx = \lim_{k\to+\infty} \frac{1}{kT+l}\int_0^{kT} f(x)dx + \lim_{k\to+\infty} \frac{1}{kT+l}\int_{kT}^{kT+l} f(x)dx$$

$$= \lim_{k\to+\infty} \frac{1}{kT+l}\int_0^{kT} f(x)dx + \lim_{k\to+\infty} \frac{1}{kT+l}\int_0^{l} f(x)dx = \lim_{k\to+\infty} \frac{1}{kT+l}\int_0^{kT} f(x)dx$$

$$= \lim_{k\to+\infty} \frac{k}{kT+l}\int_0^{T} f(x)dx = \lim_{k\to+\infty} \frac{1}{T+\frac{l}{k}}\int_0^{T} f(x)dx = \frac{1}{T}\int_0^{T} f(x)dx.$$

$$Hence,\ \lim_{a\to+\infty} \frac{1}{a}\int_0^a f(x)dx = \frac{1}{T}\int_0^{T} f(x)dx\ (a \in \mathbb{R}^+).$$

$$9.(1) \int_0^{\frac{\pi}{2}} \sin^3 x dx < \int_0^{\frac{\pi}{2}} |\sin x| \sin^2 x dx < \int_0^{\frac{\pi}{2}} \sin^2 x dx$$

$$9.(2) \int_0^1 e^{-x} dx < \int_0^1 e^{-x} e^{-x^2+x} dx = \int_0^1 e^{-x} e^{x(1-x)} dx < \int_0^1 e^{-x^2} dx$$

$$9.(3) \int_{\frac{1}{2}}^1 \sqrt{x} \ln x dx = \int_{\frac{1}{2}}^1 \sqrt[6]{x}\sqrt[3]{x} \ln x dx = \int_{\frac{1}{2}}^1 \sqrt[3]{x} \ln x dx - \int_{\frac{1}{2}}^1 \left(1-\sqrt[6]{x}\right)\sqrt[3]{x} \ln x dx > \int_{\frac{1}{2}}^1 \sqrt[3]{x} \ln x dx$$

$$10.(2) 1 \le \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx \le \frac{\pi}{2}$$

$$proof: \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx \le \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx \ge \int_0^{\frac{\pi}{2}} \frac{\frac{2}{\pi}x}{x} dx = 1$$

$$10.(4) \sqrt{2} \le \int_2^3 \sqrt[x]{x}\, dx \le \sqrt[e]{e}$$

$$let\ y = \frac{\ln x}{x}, y' = \frac{1-\ln x}{x^2} \Longrightarrow y \ge \min\{y|_{x=2}, y|_{x=3}\} = \min\left\{\frac{\ln 2}{2}, \frac{\ln 3}{3}\right\} = \frac{\ln 2}{2},\ y \le y|_{x=e} = \frac{1}{e}$$

$$\sqrt{2} = \int_2^3 e^{\frac{\ln 2}{2}} dx \le \int_2^3 \sqrt[x]{x}\, dx = \int_2^3 e^{\frac{\ln x}{x}} dx \le \int_2^3 e^{\frac{1}{e}} dx = e^{\frac{1}{e}}$$

$$11.(1)\int_0^1\sqrt{1+x^4}\,dx=\int_0^1(1+x^4)^{\frac{1}{2}}dx=\int_0^1\left(1+\frac{1}{2}x^4+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^8+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\frac{2\xi^3(x)}{\sqrt{1+\xi^4(x)}}x^{12}\right)dx,\ \xi(x)\in[0,1]$$

$$>\int_0^1\left(1+\frac{1}{2}x^4+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^8\right)dx=1.08611>1.086$$

$$\int_0^1\sqrt{1+x^4}\,dx=\int_0^1(1+x^4)^{\frac{1}{2}}dx$$

$$=\int_0^1\left(1+\frac{1}{2}x^4+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^8+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}x^{12}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{24}\frac{2\xi^3(x)}{\sqrt{1+\xi^4(x)}}x^{16}\right)dx,\ \xi(x)\in[0,1]$$

$$<\int_0^1\left(1+\frac{1}{2}x^4+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^8+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}x^{12}\right)dx=1+\frac{1}{10}-\frac{1}{72}+\frac{1}{208}=1.09092<1.097$$

$$\Longrightarrow\int_0^1\sqrt{1+x^4}\,dx\in(1.086,1.097)$$

$$11.(2)\int_0^1 e^{-x^2}dx=\int_0^1\left(1-x^2+\frac{1}{2}x^4-\frac{1}{6}x^6+\frac{e^{\xi(-x^2)}}{24}x^8\right)dx,\ \xi(-x^2)\in[-1,0]$$

$$>\int_0^1\left(1-x^2+\frac{1}{2}x^4-\frac{1}{6}x^6\right)dx=1-\frac{1}{3}+\frac{1}{10}-\frac{1}{42}=\frac{156}{210}$$

$$\int_0^1 e^{-x^2}dx=\int_0^1\left(1-x^2+\frac{1}{2}x^4-\frac{1}{6}x^6+\frac{1}{24}x^8-\frac{e^{\xi(-x^2)}}{120}x^{10}\right)dx,\ \xi(-x^2)\in[-1,0]$$

$$<\int_0^1\left(1-x^2+\frac{1}{2}x^4-\frac{1}{6}x^6+\frac{1}{24}x^8\right)dx=1-\frac{1}{3}+\frac{1}{10}-\frac{1}{42}+\frac{1}{216}=\frac{156}{210}+\frac{1}{216}<\frac{156}{210}+\frac{1}{210}=\frac{157}{210}$$

$$\Longrightarrow\frac{156}{210}<\int_0^1 e^{-x^2}dx<\frac{157}{210}.$$

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$$1.(2)\int_1^4 \frac{(x-2)^2}{x\sqrt{x}}dx=\int_1^4\left(x^{\frac{1}{2}}-4x^{-\frac{1}{2}}+4x^{-\frac{3}{2}}\right)dx=\left(\frac{2}{3}x^{\frac{3}{2}}-8x^{\frac{1}{2}}-8x^{-\frac{1}{2}}\right)\Big|_1^4=\left(\frac{16}{3}-16-4\right)-\left(\frac{2}{3}-8-8\right)$$

$$=\frac{16}{3}-16-4-\frac{2}{3}+8+8=\frac{2}{3}$$

$$1.(4)\int_2^3\frac{1+x^2}{1-x^2}dx=\int_2^3\left(\frac{2}{1-x^2}-1\right)dx=\left(\ln\left|\frac{x+1}{x-1}\right|-x\right)\Big|_2^3=\ln\frac{4}{2}-3-\ln\frac{3}{1}+2=\ln\frac{2}{3e}.$$

$$1.(6)\int_2^3(3^x-2^x)^3dx=\int_2^3(3^{3x}-3\cdot 3^{2x}2^x+3\cdot 3^x2^{2x}-2^{3x})dx$$

$$=\int_2^3(e^{3x\ln 3}-e^{(2x+1)\ln 3+x\ln 2}+e^{(x+1)\ln 3+2x\ln 2}-e^{3x\ln 2})dx$$

$$=\left(\frac{1}{3\ln 3}e^{3x\ln 3}-\frac{1}{2\ln 3+\ln 2}e^{(2x+1)\ln 3+x\ln 2}+\frac{1}{\ln 3+2\ln 2}e^{(x+1)\ln 3+2x\ln 2}-\frac{1}{3\ln 2}e^{3x\ln 2}\right)\Big|_2^3$$

$$=\frac{1}{3\ln 3}(3^9-3^6)-\frac{1}{2\ln 3+\ln 2}(3^7 2^3-3^5 2^2)+\frac{1}{\ln 3+2\ln 2}(3^4 2^6-3^3 2^4)-\frac{1}{3\ln 2}(2^9-2^6)$$

$$=\frac{3^9-3^6}{3\ln 3}-\frac{3^7 2^3-3^5 2^2}{2\ln 3+\ln 2}+\frac{3^4 2^6-3^3 2^4}{\ln 3+2\ln 2}-\frac{2^9-2^6}{3\ln 2}$$

$$1.(7)\int_1^e\frac{\ln x}{x}dx=\int_1^e\ln x d(\ln x)=\int_1^e d\left(\frac{\ln^2 x}{2}\right)=\frac{1}{2}.$$

$$1.(9)\int_{-1}^2 x|x-1|dx=\int_{-1}^1 x|x-1|dx+\int_1^2 x|x-1|dx=\int_{-1}^1 x(1-x)dx+\int_1^2 x(x-1)dx$$

$$=\int_{-1}^1 d\left(\frac{x^2}{2}-\frac{x^3}{3}\right)+\int_1^2 d\left(\frac{x^3}{3}-\frac{x^2}{2}\right)=\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{8}{3}-\frac{4}{2}\right)-\left(\frac{1}{3}-\frac{1}{2}\right)=\frac{1}{6}$$

$$1.(12)\int_0^3 f(x)dx$$

$$f(x)=\begin{cases}1-x^2, where\ 0\le x\le 1,\\ e^x-1, where\ 1<x\le 2,\\ e^{2(x-1)}-2x+3, where\ 2<x\le 3.\end{cases}$$

$$\int_0^3 f(x)dx=\int_0^1 f(x)dx+\int_1^2 f(x)dx+\int_2^3 f(x)dx$$

$$=\int_0^1(1-x^2)dx+\int_1^2(e^x-1)dx+\int_2^3(e^{2(x-1)}-2x+3)dx$$

$$=\int_0^1 d\left(x-\frac{1}{3}x^3\right)+\int_1^2 d(e^x-x)+\int_2^3 d\left(\frac{1}{2}e^{2(x-1)}-x^2+3x\right)$$

$$=\frac{2}{3}+(e^2-2)-(e-1)+\left(\frac{1}{2}e^4-9+9\right)-\left(\frac{1}{2}e^2-4+6\right)$$

$$=\frac{2}{3}+e^2-2-e+1+\frac{1}{2}e^4-\frac{1}{2}e^2-2$$

$$=\frac{1}{2}e^4+\frac{1}{2}e^2-e-\frac{7}{3}$$

$$2.(1)\begin{cases} y=\frac{1}{2}x^2 \\ x^2+y^2=8 \end{cases} \Longrightarrow x=\pm 2.$$

$$S_{upper}=\int_{-2}^{2}dx\int_{\frac{1}{2}x^2}^{\sqrt{8-x^2}}dy=\int_{-2}^{2}\left(\sqrt{8-x^2}-\frac{1}{2}x^2\right)dx=\int_{-2}^{2}\left(\sqrt{8-x^2}\right)dx-\int_{-2}^{2}\left(\frac{1}{2}x^2\right)dx$$

$$=\left(\frac{1}{2}x\sqrt{8-x^2}+4\arctan\frac{x}{\sqrt{8-x^2}}-\frac{1}{6}x^3\right)\Bigg|_{-2}^{2}$$

$$=\left(2+4\arctan 1-\frac{4}{3}\right)-\left(-2+4\arctan(-1)+\frac{4}{3}\right)=\frac{4}{3}+2\pi$$

$$S_{lower}=S-S_{upper}=8\pi-\left(\frac{4}{3}+2\pi\right)=6\pi-\frac{4}{3}$$

$$2.(3)\begin{cases} y=x+c \\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \end{cases} \Longrightarrow \frac{x^2}{a^2}+\frac{(x+c)^2}{b^2}=1 \Longrightarrow \left(\frac{1}{a^2}+\frac{1}{b^2}\right)x^2+\frac{2c}{b^2}x+\frac{c^2-b^2}{b^2}=0$$

$$\Longrightarrow (a^2+b^2)x^2+2a^2cx+a^2(c^2-b^2)=0$$

$$\Longrightarrow \Delta=4a^4c^2-4(a^2+b^2)a^2(c^2-b^2)=4a^4c^2-4(a^4c^2+a^2b^2c^2-a^2b^4-a^4b^2)$$

$$=4(b^4+a^2b^2-b^2c^2)a^2=4(b^2+a^2-c^2)a^2b^2$$

$$\Longrightarrow x_1=\frac{-2a^2c+\sqrt{\Delta}}{2(a^2+b^2)}=\frac{-2a^2c+2ab\sqrt{b^2+a^2-c^2}}{2(a^2+b^2)}=\frac{-a^2c+ab\sqrt{b^2+a^2-c^2}}{a^2+b^2}$$

$$x_2=\frac{-a^2c-ab\sqrt{b^2+a^2-c^2}}{a^2+b^2}$$

$$a^2-x_1^2=a^2-\left(\frac{-a^2c+ab\sqrt{b^2+a^2-c^2}}{a^2+b^2}\right)^2$$

$$=\left(a+\frac{-a^2c+ab\sqrt{b^2+a^2-c^2}}{a^2+b^2}\right)\left(a-\frac{-a^2c+ab\sqrt{b^2+a^2-c^2}}{a^2+b^2}\right)$$

$$=a^2\left(1+\frac{-ac+b\sqrt{b^2+a^2-c^2}}{a^2+b^2}\right)\left(1-\frac{-ac+b\sqrt{b^2+a^2-c^2}}{a^2+b^2}\right)$$

$$=a^2\frac{a^2+b^2-ac+b\sqrt{b^2+a^2-c^2}}{a^2+b^2}\frac{a^2+b^2+ac-b\sqrt{b^2+a^2-c^2}}{a^2+b^2}$$

$$=\frac{a^2\left[(a^2+b^2)^2-\left(ac-b\sqrt{b^2+a^2-c^2}\right)^2\right]}{(a^2+b^2)^2}$$

$$=\frac{a^2\left(a^4+2a^2b^2+b^4-a^2c^2+2abc\sqrt{b^2+a^2-c^2}-b^4-a^2b^2+b^2c^2\right)}{(a^2+b^2)^2}$$

$$=\frac{a^2\left(a^4+a^2b^2-a^2c^2+2abc\sqrt{b^2+a^2-c^2}+b^2c^2\right)}{(a^2+b^2)^2}$$

$$=\frac{a^2\left(a\sqrt{b^2+a^2-c^2}+bc\right)^2}{(a^2+b^2)^2}$$

$$a^2-x_2^2=a^2-\left(\frac{a^2c+ab\sqrt{b^2+a^2-c^2}}{a^2+b^2}\right)^2$$

$$=\left(a+\frac{a^2c+ab\sqrt{b^2+a^2-c^2}}{a^2+b^2}\right)\left(a-\frac{a^2c+ab\sqrt{b^2+a^2-c^2}}{a^2+b^2}\right)$$

$$=a^2\left(1+\frac{ac+b\sqrt{b^2+a^2-c^2}}{a^2+b^2}\right)\left(1-\frac{ac+b\sqrt{b^2+a^2-c^2}}{a^2+b^2}\right)$$

$$=a^2\frac{a^2+b^2+ac+b\sqrt{b^2+a^2-c^2}}{a^2+b^2}\frac{a^2+b^2-ac-b\sqrt{b^2+a^2-c^2}}{a^2+b^2}$$

$$=\frac{a^2\left[(a^2+b^2)^2-\left(ac+b\sqrt{b^2+a^2-c^2}\right)^2\right]}{(a^2+b^2)^2}$$

$$=\frac{a^2\left(a^4+2a^2b^2+b^4-a^2c^2-2abc\sqrt{b^2+a^2-c^2}-b^4-a^2b^2+b^2c^2\right)}{(a^2+b^2)^2}$$

$$=\frac{a^2\left(a^4+a^2b^2-a^2c^2-2abc\sqrt{b^2+a^2-c^2}+b^2c^2\right)}{(a^2+b^2)^2}$$

$$=\frac{a^2\left(a\sqrt{b^2+a^2-c^2}-bc\right)^2}{(a^2+b^2)^2}$$

$$\Longrightarrow x_1\sqrt{a^2-x_1^2}=a^2\frac{-ac+b\sqrt{b^2+a^2-c^2}}{a^2+b^2}\cdot\frac{a\sqrt{b^2+a^2-c^2}+bc}{a^2+b^2}$$

$$=\frac{a^2}{(a^2+b^2)^2}\left(ab(b^2+a^2-c^2)+(b^2-a^2)c\sqrt{b^2+a^2-c^2}-abc^2\right)$$

$$=\frac{a^2}{(a^2+b^2)^2}\left(ab(a^2+b^2-2c^2)+(b^2-a^2)c\sqrt{b^2+a^2-c^2}\right)$$

$$\Longrightarrow x_2\sqrt{a^2-x_2^2}=a^2\frac{-ac-b\sqrt{b^2+a^2-c^2}}{a^2+b^2}\cdot\frac{a\sqrt{b^2+a^2-c^2}-bc}{a^2+b^2}$$

$$=-\frac{a^2}{(a^2+b^2)^2}\left(ab(b^2+a^2-c^2)-(b^2-a^2)c\sqrt{b^2+a^2-c^2}-abc^2\right)$$

$$=-\frac{a^2}{(a^2+b^2)^2}\left(ab(a^2+b^2-2c^2)-(b^2-a^2)c\sqrt{b^2+a^2-c^2}\right)$$

$$\Longrightarrow\frac{x_1}{\sqrt{a^2-x_1^2}}=\frac{-ac+b\sqrt{b^2+a^2-c^2}}{a\sqrt{b^2+a^2-c^2}+bc},\frac{x_1}{\sqrt{a^2-x_1^2}}=-\frac{ac+b\sqrt{b^2+a^2-c^2}}{a\sqrt{b^2+a^2-c^2}-bc}$$

$$S_{upper}=\int_{x_2}^{x_1}dx\int_{x+c}^{b\sqrt{1-\frac{x^2}{a^2}}}dy=\int_{x_2}^{x_1}\left(\frac{b}{a}\sqrt{a^2-x^2}-x+c\right)dx$$

$$=\left(\frac{b}{2a}x\sqrt{a^2-x^2}+\frac{ab}{2}\arctan\frac{x}{\sqrt{a^2-x^2}}-\frac{1}{2}x^2+cx\right)\Bigg|_{x_2}^{x_1}$$

$$=\left(\frac{b}{2a}x_1\sqrt{a^2-x_1^2}+\frac{ab}{2}\arctan\frac{x_1}{\sqrt{a^2-x_1^2}}-\frac{1}{2}x_1^2+cx_1\right)-\left(\frac{b}{2a}x_2\sqrt{a^2-x_2^2}+\frac{ab}{2}\arctan\frac{x_2}{\sqrt{a^2-x_2^2}}-\frac{1}{2}x_2^2+cx_2\right)$$

$$=\frac{a^2b^2(b^2+a^2-c^2)}{(a^2+b^2)^2}+\frac{ab}{2}\left(\arctan\frac{-ac+b\sqrt{b^2+a^2-c^2}}{a\sqrt{b^2+a^2-c^2}+bc}+\arctan\frac{ac+b\sqrt{b^2+a^2-c^2}}{a\sqrt{b^2+a^2-c^2}-bc}\right)$$

$$-\frac{1}{2}(x_1^2-x_2^2)+c(x_1-x_2)$$

$$=\frac{a^2b^2(b^2+a^2-c^2)}{(a^2+b^2)^2}+\frac{ab}{2}\left(\arctan\frac{-ac+b\sqrt{b^2+a^2-c^2}}{a\sqrt{b^2+a^2-c^2}+bc}+\arctan\frac{ac+b\sqrt{b^2+a^2-c^2}}{a\sqrt{b^2+a^2-c^2}-bc}\right)$$

$$\frac{2a^2b^2(b^2+a^2-c^2)}{(a^2+b^2)^2}+\frac{2abc\sqrt{b^2+a^2-c^2}}{a^2+b^2}$$

$$=\frac{3a^2b^2(b^2+a^2-c^2)}{(a^2+b^2)^2}+\frac{ab}{2}\left(\arctan\frac{-ac+b\sqrt{b^2+a^2-c^2}}{a\sqrt{b^2+a^2-c^2}+bc}+\arctan\frac{ac+b\sqrt{b^2+a^2-c^2}}{a\sqrt{b^2+a^2-c^2}-bc}\right)+\frac{2abc\sqrt{b^2+a^2-c^2}}{a^2+b^2}$$

$$4.(2)\int_{\sqrt{2}}^{2}\frac{1}{x\sqrt{x^2-1}}dx$$

$$\int\frac{1}{x\sqrt{x^2-1}}dx\left(let\ p=x,q=\sqrt{x^2-1},p^2-q^2=1\right)$$

$$=\int\frac{1}{pq}dp=\int\frac{1}{p^2}dq=\int\frac{1}{q^2+1}dq=\arctan q=\arctan\sqrt{x^2-1}$$

$$then\ \int_{\sqrt{2}}^{2}\frac{1}{x\sqrt{x^2-1}}dx=\left(\arctan\sqrt{x^2-1}\right)\Big|_{\sqrt{2}}^{2}=\arctan\sqrt{3}-\arctan 1=\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}.$$

$$4.(4)\int_{-\ln 2}^{\ln 2}\frac{dx}{e^x+e^{-x}}=\int_{-\ln 2}^{\ln 2}\frac{e^x dx}{e^{2x}+1}=\int_{-\ln 2}^{\ln 2}\frac{de^x}{e^{2x}+1}=\arctan e^x|_{-\ln 2}^{\ln 2}=\arctan 2-\arctan\frac{1}{2}=\arctan\frac{3}{4}.$$

$$4.(6)\int_0^{\frac{\pi}{6}}\frac{dx}{\cos^3 x}\overset{t=\cos x}{=}\int_1^{\frac{\sqrt{3}}{2}}\frac{d\arccos t}{t^3}=\int_1^{\frac{\sqrt{3}}{2}}\frac{1}{t^3}\left(-\frac{1}{\sqrt{1-t^2}}\right)dt=\int_{\frac{\sqrt{3}}{2}}^{1}\frac{1}{t^3}\frac{1}{\sqrt{1-t^2}}dt$$

$$\int\frac{1}{t^3}\frac{1}{\sqrt{1-t^2}}dt\left(let\ p=t,q=\sqrt{1-t^2},p^2+q^2=1\right)$$

$$=\int\frac{1}{p^3q}dp=\int\frac{1}{p^4}dq=\int\frac{1}{(1-q^2)^2}dq=-\frac{1}{2}\frac{q}{q^2-1}+\frac{1}{4}\ln\left|\frac{q+1}{q-1}\right|$$

$$=\frac{\sqrt{1-t^2}}{2t^2}+\frac{1}{4}\ln\left|\frac{\sqrt{1-t^2}+1}{\sqrt{1-t^2}-1}\right|=\frac{\sin x}{2\cos^2 x}+\frac{1}{4}\ln\left|\frac{\sin x+1}{\sin x-1}\right|$$

$$\Longrightarrow\int_0^{\frac{\pi}{6}}\frac{dx}{\cos^3 x}=\left(\frac{\sin x}{2\cos^2 x}+\frac{1}{4}\ln\left|\frac{\sin x+1}{\sin x-1}\right|\right)\Big|_0^{\frac{\pi}{6}}=\frac{1}{3}+\frac{\ln 3}{4}$$

$$4.(8)\int_0^{\frac{\sqrt{3}}{2}}\frac{x^3dx}{\sqrt{1-x^2}}\overset{x=\sin t}{=}\int_0^{\frac{\pi}{3}}\frac{\sin^3 t d\sin t}{\cos t}=\int_0^{\frac{\pi}{3}}\sin^3 t dt=-\int_0^{\frac{\pi}{3}}\sin^2 t d\cos t$$

$$=-\int_0^{\frac{\pi}{3}}(1-\cos^2 t)d\cos t=-\int_0^{\frac{\pi}{3}}d\cos t+\int_0^{\frac{\pi}{3}}d\frac{\cos^3 t}{3}=\frac{1}{24}-\frac{1}{3}-\frac{1}{2}+1=\frac{5}{24}$$

$$4.(9)\int_0^{\frac{\sqrt{3}}{2}}\frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}}dx\overset{\sqrt{x}=\sin t}{=}\int_0^{\arcsin\sqrt{\frac{\sqrt{3}}{2}}}\frac{t}{\sin t\cos t}d\sin^2 t=2\int_0^{\arcsin\sqrt{\frac{\sqrt{3}}{2}}}tdt=\arcsin^2\sqrt{\frac{\sqrt{3}}{2}}.$$

$$4.(10)\int_e^{e^2}\frac{\ln x}{x\sqrt{1+\ln x}}dx=\int_e^{e^2}\frac{\ln x}{\sqrt{1+\ln x}}d(\ln x)\overset{t=\ln x}{=}\int_1^2\frac{t}{\sqrt{1+t}}dt=2\int_1^2 td\sqrt{1+t}$$

$$=2\left(t\sqrt{1+t^2}\Big|_1^2-\int_1^2\sqrt{1+t}\,d(1+t)\right)=2\left(2\sqrt{5}-\sqrt{2}-\frac{2}{3}\int_1^2 d(1+t)^{\frac{3}{2}}\right)$$

$$=4\sqrt{5}-2\sqrt{2}-\frac{4}{3}\left(3\sqrt{3}-2\sqrt{2}\right)=4\sqrt{5}-4\sqrt{3}+\frac{2}{3}\sqrt{2}$$

$$5.(1)\int_0^{\ln 2} xe^{-x}dx=-\int_0^{\ln 2} xde^{-x}=-\left(xe^{-x}\Big|_0^{\ln 2}-\int_0^{\ln 2} e^{-x}dx\right)=-xe^{-x}\Big|_0^{\ln 2}-\int_0^{\ln 2} de^{-x}=-\frac{\ln 2}{2}+\frac{1}{2}$$

$$5.(3)\int_0^{\pi} x^3\sin x dx=-\int_0^{\pi} x^3 d\cos x=-\left(x^3\cos x\Big|_0^{\pi}-\int_0^{\pi}\cos x dx^3\right)=-\left(-\pi^3-3\int_0^{\pi}x^2\cos x dx\right)$$

$$=\pi^3+3\int_0^{\pi}x^2\cos x dx=\pi^3+3\int_0^{\pi}x^2 d\sin x=\pi^3+3\left(x^2\sin x\Big|_0^{\pi}-\int_0^{\pi}\sin x dx^2\right)$$

$$=\pi^3-3\int_0^{\pi}\sin x dx^2=\pi^3-6\int_0^{\pi}x\sin x dx=\pi^3+6\int_0^{\pi}x d\cos x=\pi^3+6\left(x\cos x\Big|_0^{\pi}-\int_0^{\pi}\cos x dx\right)$$

$$=\pi^3-6\pi-12$$

$$5.(5)\int_0^{\sqrt{3}} x\arctan x dx=\frac{1}{2}\int_0^{\sqrt{3}}\arctan x dx^2=\frac{1}{2}\left(x^2\arctan x\Big|_0^{\sqrt{3}}-\int_0^{\sqrt{3}}x^2 d\arctan x\right)$$

$$=\frac{1}{2}\left(\pi-\int_0^{\sqrt{3}}\frac{x^2}{1+x^2}dx\right)=\frac{\pi}{2}-\frac{1}{2}\int_0^{\sqrt{3}}\frac{1+x^2-1}{1+x^2}dx=\frac{\pi}{2}-\frac{1}{2}\int_0^{\sqrt{3}}\left(1-\frac{1}{1+x^2}\right)dx$$

$$=\frac{\pi}{2}-\frac{1}{2}(x-\arctan x)\Big|_0^{\sqrt{3}}=\frac{\pi}{2}-\frac{1}{2}\left(\sqrt{3}-\frac{\pi}{3}\right)=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}.$$

$$5.(8)\int_0^1\ln\left(x+\sqrt{1+x^2}\right)dx\underset{x=\frac{t^2-1}{2t}}{\overset{t=x+\sqrt{1+x^2}}{=}}\frac{1}{2}\int_0^1\ln t d\left(t-\frac{1}{t}\right)=\frac{1}{2}\left(t-\frac{1}{t}\right)\ln t\Big|_1^{1+\sqrt{2}}-\frac{1}{2}\int_1^{1+\sqrt{2}}\left(t-\frac{1}{t}\right)d\ln t$$

$$=\frac{1}{2}\left((1+\sqrt{2})-\frac{1}{1+\sqrt{2}}\right)\ln(1+\sqrt{2})-\frac{1}{2}\int_1^{1+\sqrt{2}}\left(1-\frac{1}{t^2}\right)dt=\frac{1}{2}\left((1+\sqrt{2})-(\sqrt{2}-1)\right)\ln(1+\sqrt{2})-\frac{1}{2}\int_1^{1+\sqrt{2}}\left(1-\frac{1}{t^2}\right)dt$$

$$=\ln(1+\sqrt{2})-\frac{1}{2}\int_1^{1+\sqrt{2}}d\left(t+\frac{1}{t}\right)=\ln(1+\sqrt{2})-\frac{1}{2}\left((1+\sqrt{2})+\frac{1}{1+\sqrt{2}}\right)+\frac{1}{2}2=\ln(1+\sqrt{2})-\sqrt{2}+1$$

$$6.(2)\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2}=\int_0^1\frac{1}{1+x^2}dx=\arctan x\Big|_0^1=\frac{\pi}{4}$$

$$6.(4)\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\cdots+\sqrt{1+\frac{n}{n}}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{1+\frac{k}{n}}=\int_0^1\sqrt{1+x}\,dx=\frac{2}{3}(1+x)^{\frac{3}{2}}\Big|_0^1$$

$$=\frac{2}{3}(2\sqrt{2}-1)=\frac{4\sqrt{2}}{3}-\frac{2}{3}.$$