11/20 homework

1.(2) $y=x^3, y'=3x^2, y''=6x \Rightarrow y=x^3$是凸函数，当且仅当$x>0$, $y=x^3$是凹函数，当且仅当$x<0$

1.(4) $y=e^{-x^2}, y'=-2xe^{-x^2}, y''=(4x^2-2)e^{-x^2}$

$\Rightarrow y=e^{-x^2}$是凸函数，当且仅当$x>\frac{\sqrt{2}}{2} \vee x<-\frac{\sqrt{2}}{2}$, $y=e^{-x^2}$是凹函数，当且仅当$-\frac{\sqrt{2}}{2}<x<\frac{\sqrt{2}}{2}$

1.(6) $y=\frac{1+x}{1+x^2}, y'=\frac{1+x^2-2x(1+x)}{(1+x^2)^2}=\frac{1-2x-x^2}{(1+x^2)^2}, y''=\frac{(-2-2x)(1+x^2)-(1-2x-x^2)4x}{(1+x^2)^3}$

$=\frac{-2-6x+6x^2+2x^3}{(1+x^2)^3}=\frac{2(x^3+3x^2-3x-1)}{(1+x^2)^3}=\frac{2(x-1)(x^2+4x+1)}{(1+x^2)^3}$

$\Rightarrow y=\frac{1+x}{1+x^2}$是凸函数，当且仅当$x\in\left(-2-\sqrt{3},-2+\sqrt{3}\right)\cup(1,+\infty)$,

$y=\frac{1+x}{1+x^2}$是凹函数，当且仅当$x\in\left(-\infty,-2-\sqrt{3}\right)\cup\left(-2+\sqrt{3},1\right)$

2.(2) $f(x):=x^p, f'(x)=px^{p-1}, f''(x)=p(p-1)x^{p-2}<0 \Rightarrow f(x)$是凹函数

$\Rightarrow \frac{f(a)+f(b)}{2}<f\left(\frac{a+b}{2}\right), i.e.\ \frac{a^p+b^p}{2}<\left(\frac{a+b}{2}\right)^p$

2.(3) $f(x):=e^x, f'(x)=e^x, f''(x)=e^x>0 \Rightarrow f(x)$是凸函数

$\Rightarrow \frac{f(a)+f(b)}{2}<f\left(\frac{a+b}{2}\right), i.e. \frac{e^a+e^b}{2}>e^{\frac{a+b}{2}}$

3.(2) $proof$: f,g是凸函数, $f,g\geq 0$, f,g单调递增,下证明：$f\cdot g$也是凸函数

不妨设$f,g\in C^2$,否则使用现代磨光技术.

$f'',g''\geq 0, (f\cdot g)'=f'\cdot g+f\cdot g', (f\cdot g)''=(f'\cdot g+f\cdot g')'=f''\cdot g+2f'\cdot g'+f\cdot g''\geq 0$,因此$f\cdot g$也是凸函数

3.(4) $proof$:

$g\circ f(\lambda x_1+(1-\lambda)x_2)=g\left(f(\lambda x_1+(1-\lambda)x_2)\right)\geq g(\lambda f(x_1)+(1-\lambda)f(x_2))\geq \lambda g\circ f(x_1)+(1-\lambda)g\circ f(x_2)$

因此$f\circ g$也是凸函数

5.$proof$:不妨设$f(x)$二阶可导，否则使用现代磨光技术.

("$\Rightarrow$"): $f''(\xi)\geq 0, \forall \xi\in(a,b) \Rightarrow f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\xi)}{2}(x-x_0)^2\geq f(x_0)+f'(x_0)(x-x_0)$

("$\Leftarrow$"): $f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\xi)}{2}(x-x_0)^2\geq f(x_0)+f'(x_0)(x-x_0), \forall x_0,x\in(a,b)$

$\Rightarrow \xi\in(a,b) \Rightarrow \forall \xi\in(a,b), f''(\xi)\geq 0 \Rightarrow f$是凸函数

6.$proof$:不妨设$f(x)$二阶可导，否则使用现代磨光技术.

$\left(\frac{f(x)}{x}\right)'=\frac{xf'(x)-f(x)}{x^2}$

$f(x_0)=f(x)+f'(x)(x_0-x)+\frac{f''(\xi)}{2}(x-x_0)^2\geq f(x)+f'(x)(x_0-x), let\ x_0=0$

$0\geq f(x)+f'(x)(-x) \Rightarrow xf'(x)-f(x)\geq 0 \Rightarrow \left(\frac{f(x)}{x}\right)'\geq 0 \Rightarrow \frac{f(x)}{x}$单调递增.

显然f严格上凸说明取不到等号，故$\frac{f(x)}{x}$严格单调递增.

11/21/2023 homework

$$1.(1)\ y=\frac{1-x}{\sqrt{1+x}}$$

$$=1-\frac{3}{2}x+\frac{7}{8}x^2-\frac{11}{16}x^3+\frac{75}{128}x^4-\frac{133}{256}(\xi x)^5,\xi\in(0,1)$$

$$1.(3)\ y=\ln(1+e^x)$$

$$=\ln 2+\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{192}x^4+\frac{1}{2880}(\xi x)^6,\xi\in(0,1)$$

$$1.(4)\ y=e^{\sin x}$$

$$=1+x+\frac{x^2}{2}-\frac{x^4}{8}-\frac{1}{15}(\xi x)^5,\xi\in(0,1)$$

$$2.(3)\ y=e^{2x-x^2}$$

$$=1+2x+x^2-\frac{2}{3}x^3-\frac{5}{6}x^4-\frac{1}{15}x^5+o(x^5)$$

$$2.(4)\ y=\ln\cos x$$

$$=-\frac{1}{2}x^2-\frac{1}{12}x^4+o(x^5)$$

$$4.(1)\ \lim_{x\to 0}\frac{1}{x}\left(\frac{1}{\tan x}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\frac{\cos x-1}{x\sin x}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{2}x^2+o(x^3)}{x(x+o(x^2))}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{2}x^2+o(x^3)}{x^2+o(x^3)}$$

$$=-\frac{1}{2}$$

$$4.(3)\ \lim_{x\to 0}\frac{\cos 2x-e^{-x^2}}{x^4}=\lim_{x\to 0}\frac{\left(1-\frac{1}{2}(2x)^2+\frac{1}{24}(2x)^4+o(x^5)\right)-(1-2x^2+2x^4+o(x^5))}{x^4}$$

$$=\lim_{x\to 0}\frac{-\frac{4}{3}x^4+o(x^5)}{x^4}$$

$$=-\frac{4}{3}$$

$$4.(4)\ \lim_{x\to 0}\frac{e^x\sin x-x(1+x)}{\sin^3 x}=\lim_{x\to 0}\frac{\left(1+x+\frac{1}{2}x^2+o(x^2)\right)(x+o(x^2))-x(1+x)}{(x+o(x^2))^3}$$

$$=\lim_{x\to 0}\frac{\left(1+x+\frac{1}{2}x^2+o(x^2)\right)\left(x-\frac{1}{6}x^3+o(x^4)\right)-x(1+x)}{x^3+o(x^3)}$$

$$=\lim_{x\to 0}\frac{\frac{1}{3}x^3+o(x^3)}{x^3+o(x^3)}$$

$$=\frac{1}{3}$$

$$4.(6)\ \lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right)$$

$$=\lim_{x\to 0}\frac{x-\ln(1+x)}{x\ln(1+x)}$$

$$=\lim_{x\to 0}\frac{x-\left(x-\frac{1}{2}x^2+o(x^2)\right)}{x\left(x-\frac{1}{2}x^2+o(x^2)\right)}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2}x^2+o(x^2)}{x^2+o(x^2)}$$

$$=\frac{1}{2}$$

$$6.(1)\left(\frac{\sin x}{x}\right)^3-\cos x=\left(\frac{x-\frac{1}{6}x^3+\frac{1}{120}x^5+o(x^6)}{x}\right)^3-\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+o(x^5)\right)$$

$$=\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4+o(x^5)\right)^3-\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+o(x^5)\right)$$

$$=\left(1-\frac{1}{2}x^2+\frac{13}{120}x^4+o(x^5)\right)-\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+o(x^5)\right)$$

$$=\frac{1}{15}x^4+o(x^5)$$

$$6.(3)\ (1+x)^{\frac{1}{x}}-e=e^{\frac{1}{x}\ln(1+x)}-e$$

$$=e^{\frac{1}{x}\left(x-\frac{1}{2}x^2+o(x^2)\right)}-e$$

$$=e^{1-\frac{1}{2}x+o(x)}-e$$

$$=e\left(e^{-\frac{1}{2}x+o(x)}-1\right)$$

$$=e\left(1-\frac{1}{2}x+o(x)+o\left(-\frac{1}{2}x+o(x)\right)-1\right)$$

$$=-\frac{e}{2}x+o(x)$$

$$6.(4)1-\frac{(1+x)^{\frac{1}{x}}}{e}=1-\frac{e^{\frac{1}{x}\ln(1+x)}}{e}$$

$$=1-e^{\frac{1}{x}\ln(1+x)-1}$$

$$=1-e^{\frac{1}{x}\left(x-\frac{1}{2}x^2+o(x^2)\right)-1}$$

$$=1-e^{1-\frac{1}{2}x+o(x)-1}$$

$$=1-e^{-\frac{1}{2}x+o(x)}$$

$$=1-\left(1-\frac{1}{2}x+o(x)+o\left(-\frac{1}{2}x+o(x)\right)\right)$$

$$=\frac{1}{2}x+o(x)$$

11/23/2023 homework

$$1.(3)f(x)=x\ln x-1,f'(x)=1+\ln x,f''(x)=\frac{1}{x}>0,f(e)=e-1>0,$$

$$let\ \{x_n\}_{n=1}^{\infty}\to \text{the root of } f(x):=\hat{x},\ \text{obviously } f(x) \text{ has only one root.}$$

$$\text{pick } x_1=e,\ \text{then } x_2=x_1-\frac{f(x_1)}{f'(x_1)}=\frac{e+1}{2},x_3=x_2-\frac{f(x_2)}{f'(x_2)}=\frac{3+e}{2\left(1+\ln\left(\frac{1+e}{2}\right)\right)}=1.76478$$

$$x_4=x_3-\frac{f(x_3)}{f'(x_3)}=\frac{5+e+2\ln\left(\frac{1+e}{2}\right)}{2\left(1+\ln\left(\frac{1+e}{2}\right)\right)\left(1+\ln\left(\frac{3+e}{2\left(1+\ln\left(\frac{1+e}{2}\right)\right)}\right)\right)}=1.76322 \text{ is the approximation of } \hat{x}.$$

$$1.(5)f(x)=x^4-10x^3+1,f'(x)=4x^3-30x^2,f''(x)=12x^2-60x$$

$$\text{obviously } f(x) \text{ has only } two \text{ root}$$

$$let\ \{x_n\}_{n=1}^{\infty}\to \text{the root of } f(x):=\hat{x},\ let\ \{y_n\}_{n=1}^{\infty}\to \text{the root of } f(x):=\hat{y}$$

$$f(0)=1,f(1)=-8,f(9)=-728,f(10)=1$$

$$f''(0)=0,f''(1)=-48,f''(9)=432,f''(10)=600$$

$$\text{pick } x_1=1,y_1=10$$

$$\text{then } x_2=x_1-\frac{f(x_1)}{f'(x_1)}=\frac{9}{13}\approx 0.692308,x_3=x_2-\frac{f(x_2)}{f'(x_2)}=\frac{99209}{186381}\approx 0.532291$$

$$x_4=x_3-\frac{f(x_3)}{f'(x_3)}=\frac{759326917107894697603}{1588196157269159934639}\approx 0.478107$$

$$x_5=x_4-\frac{f(x_4)}{f'(x_4)}\approx 0.471778,x_6=x_5-\frac{f(x_5)}{f'(x_5)}\approx 0.471695 \text{ is the approximation of } \hat{x}.$$

$$y_2=y_1-\frac{f(y_1)}{f'(y_1)}=\frac{9999}{1000}=9.999,y_3=y_2-\frac{f(y_2)}{f'(y_2)}=\frac{9993001199900003}{999400089996000}=9.999 \text{ is the approximation of } \hat{x}.$$

$$2.proof:$$

$$f'(x)\neq 0\ wherever\ x\in[c,d]\Rightarrow \text{without loss of generrality: assume that } f'(x)>0,\text{wherever } x\in[c,d]\ ,f(c)<0<f(d)$$

$$we\ denote\ the\ root\ of\ f(x)\ as\ \hat{x},thus\ f(x)<0\ iff\ x<\hat{x}$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$0=f(\hat{x})=f(x_n)+f'(x_n)(\hat{x}-x_n)+\frac{f''(\xi_n)}{2}(\hat{x}-x_n)^2,\xi_n\ is\ between\ \hat{x}\ and\ x_n$$

$$|x_{n+1}-\hat{x}|=\left|x_n-\frac{f(x_n)}{f'(x_n)}-\hat{x}\right|=\left|\frac{f''(\xi_n)}{2f'(x_n)}(\hat{x}-x_n)^2\right|\leq\frac{|f''(\xi_n)|}{2|f'(x_n)|}(\hat{x}-x_n)^2\leq\sup_{n\in[c,d]}\frac{|f''(\xi_n)|}{2|f'(x_n)|}(\hat{x}-x_n)^2$$

$$\leq\frac{1}{2}\frac{\sup\limits_{x\in[c,d]}|f''(x)|}{\inf\limits_{x\in[c,d]}|f'(x)|}\left|\frac{2a+b}{3}-\frac{a+2b}{3}\right|(\hat{x}-x_n)=\frac{1}{6}\frac{\sup\limits_{x\in[c,d]}|f''(x)|}{\inf\limits_{x\in[c,d]}|f'(x)|}|b-a|\,|\hat{x}-x_n|=q|\hat{x}-x_n|\leq\cdots\leq q^n|\hat{x}-x_1|\to 0$$

$$Hence,\{x_n\}_{n=1}^{\infty}\to\hat{x}.$$

$$3.proof:$$

without loss of generrality: assume that $f(a)<0<f(b)$

we denote the root of $f(x)$ *as* $\hat{x}$, *thus* $f(x)<0$ *iff* $x<\hat{x}$

$$x_{n+1}=x_n-\theta cf(x_n),\left(0<c<\frac{1}{L}\right)(\theta=1\ iff\ f'>0,\ otherwise\ \theta=-1)$$

$$L>0\Longrightarrow f'(x)\neq 0,\forall x\in[a,b]$$

without loss of generrality: assume that $f'(x)>0,\forall x\in[a,b]$

$$then\ x_{n+1}=x_n-cf(x_n)$$

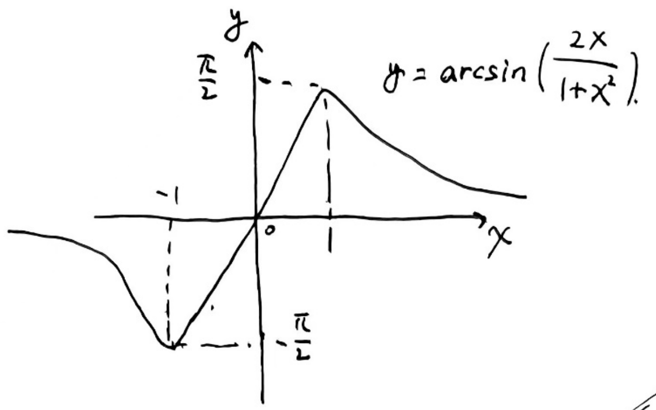
$$0=f(\hat{x})=f(x_n)+f'(\xi_n)(\hat{x}-x_\mathrm{n}),\xi_n \text{ is between } \hat{x} \text{ and } x_\mathrm{n}$$

$$|x_{n+1}-\hat{x}|=|x_n-cf(x_n)-\hat{x}|=|x_n+c(f'(\xi_n)(\hat{x}-x_\mathrm{n}))-\hat{x}|=|1-cf'(\xi_n)||\hat{x}-x_\mathrm{n}|$$

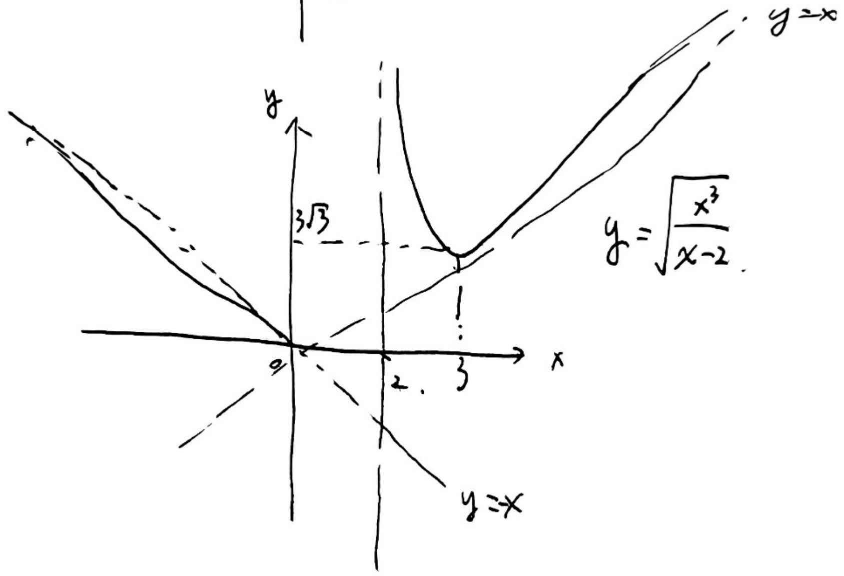
$$0<c<\frac{1}{L}=\frac{1}{\inf\limits_{a\leq x\leq b}|f'(x)|}\leq\frac{1}{|f'(x)|}=\frac{1}{f'(x)},\forall a\leq x\leq b\overset{\xi_n\in[a,b]}{\Longrightarrow}1-cf'(\xi_n)\in(0,1)$$

$$\Longrightarrow|\hat{x}-x_\mathrm{n}|=|1-cf'(\xi_n)|^{n-1}|\hat{x}-x_1|\rightarrow 0$$

Hence, $\{x_n\}_{n=1}^{\infty}$ *converges at* $\hat{x}$.



y
π/2
y = arcsin(2x/(1+x²))
-1
o
x
-π/2



y = x
y
3√3
y = √(x³/(x-2))
o
2
3
x
y = -x