11/13 homework

$$1(1)\lim_{x\to0}\frac{x\cot x-1}{x^2}=\lim_{x\to0}\frac{x-\tan x}{x^2\tan x}=\lim_{x\to0}\frac{x-\tan x}{\frac{1}{3}x^3}\frac{\frac{1}{3}x^3}{x^2\tan x}=\frac{1}{3}\lim_{x\to0}\frac{x-\tan x}{\frac{1}{3}x^3}$$

$$=\frac{1}{3}\lim_{x\to0}\frac{1-\frac{1}{\cos^2 x}}{x^2}=\frac{1}{3}\lim_{x\to0}\frac{\cos^2 x-1}{x^2\cos^2 x}=\frac{2}{3}\lim_{x\to0}\frac{\cos x-1}{x^2}=\frac{2}{3}\lim_{x\to0}\frac{-\sin x}{2x}=-\frac{1}{3}$$

$$1(2)\lim_{x\to0}\frac{1-\cos x^2}{x^2\sin x^2}=\lim_{x\to0^+}\frac{1-\cos x}{x\sin x}=\lim_{x\to0^+}\frac{\sin x}{x\cos x+\sin x}=\lim_{x\to0^+}\frac{1}{\frac{x}{\tan x}+1}=\frac{1}{\lim\limits_{x\to0^+}\frac{x}{\tan x}+1}$$

$$=\frac{1}{\lim\limits_{x\to0^+}\frac{1}{\frac{1}{\cos^2 x}}+1}=\frac{1}{\lim\limits_{x\to0^+}\cos^2 x+1}=\frac{1}{2}$$

$$1(3)\lim_{x\to0}\frac{\arcsin 2x-2\arcsin x}{x^2\sin x}=\lim_{x\to0}\frac{\arcsin 2x-2\arcsin x}{x^3}\frac{x^3}{x^2\sin x}=\lim_{x\to0}\frac{\arcsin 2x-2\arcsin x}{x^3}$$

$$=\lim_{x\to0}\frac{\frac{2}{\sqrt{1-4x^2}}-\frac{2}{\sqrt{1-x^2}}}{3x^2}=\frac{2}{3}\lim_{x\to0}\frac{\sqrt{1-x^2}-\sqrt{1-4x^2}}{x^2\sqrt{1-4x^2}\sqrt{1-x^2}}$$

$$=\frac{2}{3}\lim_{x\to0}\frac{3x^2}{x^2\sqrt{1-4x^2}\sqrt{1-x^2}\left(\sqrt{1-x^2}+\sqrt{1-4x^2}\right)}=\lim_{x\to0}\frac{2}{\sqrt{1-4x^2}\sqrt{1-x^2}\left(\sqrt{1-x^2}+\sqrt{1-4x^2}\right)}=1$$

$$1(5)\lim_{x\to0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to0}\frac{e^x-1-x}{x\left(e^x-1\right)}=\lim_{x\to0}\frac{e^x-1-x}{x^2}\frac{x^2}{x\left(e^x-1\right)}=\lim_{x\to0}\frac{e^x-1-x}{x^2}$$

$$=\lim_{x\to0}\frac{e^x-1}{2x}=\lim_{x\to0}\frac{e^x}{2}=\frac{1}{2}$$

$$1(6)\lim_{x\to0}\left(\frac{\cot x}{x}-\csc^2 x\right)=\lim_{x\to0}\left(\frac{\cos x}{x\sin x}-\frac{1}{\sin^2 x}\right)=\lim_{x\to0}\frac{\sin x\cos x-x}{x\sin^2 x}=\lim_{x\to0}\frac{\frac{1}{2}\sin 2x-x}{x\sin^2 x}$$

$$=\lim_{x\to0}\frac{\frac{1}{2}\sin 2x-x}{x^3}\frac{x^3}{x\sin^2 x}=\lim_{x\to0}\frac{\cos 2x-1}{3x^2}=\lim_{x\to0}\frac{-2\sin 2x}{6x}=-\frac{2}{3}$$

$$2(1)\lim_{x\to 0} x^{x^2}=e^{\lim\limits_{x\to 0} x^2\ln x}=e^{\lim\limits_{x\to 0}\frac{\ln x}{\frac{1}{x^2}}}=e^{\lim\limits_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x}\frac{2}{x^3}}}=e^{\lim\limits_{x\to 0}-\frac{x^4}{2}}=1$$

$$2(4)\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan(2x)}=\lim_{x\to\frac{\pi}{4}}(\tan x)^{\frac{2\tan x}{1-\tan^2 x}}=\lim_{x\to 1} x^{\frac{2x}{1-x^2}}=e^{\lim\limits_{x\to 1}\frac{2x\ln x}{1-x^2}}=e^{\lim\limits_{x\to 1}\frac{2+2\ln x}{-2x}}=e^{-1}=\frac{1}{e}$$

$$2(5)\lim_{x\to 0}\left(\frac{\arcsin x}{x}\right)^{\frac{1}{x^2}}=e^{\lim\limits_{x\to 0}\frac{\ln\left(\frac{\arcsin x}{x}\right)}{x^2}}=e^{\lim\limits_{x\to 0}\frac{\ln\left(1+\frac{\arcsin x-x}{x}\right)}{\frac{\arcsin x-x}{x}}\frac{\frac{\arcsin x-x}{x}}{x^2}}=e^{\lim\limits_{x\to 0}\frac{\ln\left(1+\frac{\arcsin x-x}{x}\right)}{\frac{\arcsin x-x}{x}}\frac{\arcsin x-x}{x^3}}=e^{\lim\limits_{x\to 0}\frac{\ln\left(1+\frac{\arcsin x-x}{x}\right)}{\frac{\arcsin x-x}{x}}\lim\limits_{x\to 0}\frac{\arcsin x-x}{x^3}}$$

$$\lim_{x\to 0}\frac{\arcsin x-x}{x}=0,\lim_{x\to 0}\frac{\arcsin x-x}{x^3}=\lim_{x\to 0}\frac{\frac{1}{\sqrt{1-x^2}}-1}{3x^2}=\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{3x^2\sqrt{1-x^2}}$$

$$=\lim_{x\to 0}\frac{x^2}{3x^2\sqrt{1-x^2}\left(1+\sqrt{1-x^2}\right)}=\frac{1}{6}$$

$$\Rightarrow\lim_{x\to 0}\left(\frac{\arcsin x}{x}\right)^{\frac{1}{x^2}}=e^{\lim\limits_{x\to 0}\frac{\ln\left(1+\frac{\arcsin x-x}{x}\right)}{\frac{\arcsin x-x}{x}}\lim\limits_{x\to 0}\frac{\arcsin x-x}{x^3}}=e^{\lim\limits_{x\to 0}\frac{\ln\left(1+\frac{\arcsin x-x}{x}\right)}{\frac{\arcsin x-x}{x}}\lim\limits_{x\to 0}\frac{\arcsin x-x}{x^3}}=e^{\frac{1}{6}}$$

$$2(6)\lim_{x\to+\infty}\left(\frac{2}{\pi}\arctan x\right)^{x}=e^{\lim\limits_{x\to+\infty} x\ln\left(\frac{2}{\pi}\arctan x\right)}=e^{\lim\limits_{x\to\frac{\pi}{2}^-}\tan x\ln\left(\frac{2}{\pi}x\right)}=e^{\lim\limits_{x\to\frac{\pi}{2}^-}\frac{\ln\left(\frac{2}{\pi}x\right)}{\cot x}}=e^{\lim\limits_{x\to\frac{\pi}{2}^-}-\frac{\frac{1}{x}}{\frac{1}{\sin^2 x}}}$$

$$=e^{\lim\limits_{x\to\frac{\pi}{2}^-}-\frac{\sin^2 x}{x}}=e^{-\frac{2}{\pi}}$$

$$3(2)\lim_{n\to\infty} n^2\left(\sqrt[n]{a}-\sqrt[n+1]{a}\right)=\lim_{n\to\infty} n^2\left(a^{\frac{1}{n}}-a^{\frac{1}{n+1}}\right)=\lim_{n\to\infty} a^{\frac{1}{n+1}}n^2\left(a^{\frac{1}{n(n+1)}}-1\right)=\lim_{x\to 0}\frac{a^x-1}{x}$$

$$=\lim_{x\to 0} a^x\ln a=\ln a$$

$$3(4)\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^{n}=e^{\lim\limits_{n\to\infty} n\ln\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}}=e^{\lim\limits_{x\to 0}\frac{\ln\frac{a^x+b^x}{2}}{x}}=e^{\lim\limits_{x\to 0}\frac{a^x\ln a+b^x\ln b}{a^x+b^x}}=e^{\frac{\ln a+\ln b}{2}}=\sqrt{ab}$$

$$3(6)\lim_{n\to\infty}\left(\frac{\cos\frac{\pi}{n}}{\cosh\frac{\pi}{n}}\right)^{n^2}=e^{\lim\limits_{n\to\infty} n^2\ln\frac{\cos\frac{\pi}{n}}{\cosh\frac{\pi}{n}}}=e^{\pi^2\lim\limits_{x\to 0}\frac{\ln\frac{\cos x}{\cosh x}}{x^2}}=e^{\pi^2\lim\limits_{x\to 0}\frac{\ln\frac{2\cos x}{e^x+e^{-x}}}{x^2}}=e^{\pi^2\lim\limits_{x\to 0}\frac{\ln\cos x-\ln\left(e^x+e^{-x}\right)+\ln 2}{x^2}}$$

$$=e^{\pi^2\lim\limits_{x\to 0}\frac{-\tan x-\frac{e^x-e^{-x}}{e^x+e^{-x}}}{2x}}=e^{\pi^2\lim\limits_{x\to 0}\frac{-\tan x-1+\frac{2e^{-x}}{e^x+e^{-x}}}{2x}}=e^{\pi^2\lim\limits_{x\to 0}\frac{-\tan x-1+\frac{2}{e^{2x}+1}}{2x}}=e^{\pi^2\lim\limits_{x\to 0}\frac{-\frac{1}{\cos^2 x}-\frac{4e^{2x}}{\left(e^{2x}+1\right)^2}}{2}}=e^{-\pi^2}$$

$$3(8)\lim_{n\to\infty}\tan^n\left(\frac{\pi}{4}+\frac{1}{n}\right)=\lim_{n\to\infty}\left(\frac{1+\tan\frac{1}{n}}{1-\tan\frac{1}{n}}\right)^{n}=\lim_{n\to\infty}\left(1+\frac{2\tan\frac{1}{n}}{1-\tan\frac{1}{n}}\right)^{n}=e^{\lim\limits_{n\to\infty} n\ln\left(1+\frac{2\tan\frac{1}{n}}{1-\tan\frac{1}{n}}\right)}$$

$$=e^{\lim\limits_{x\to 0}\frac{\ln\left(1+\frac{2\tan x}{1-\tan x}\right)}{x}}=e^{\lim\limits_{x\to 0}\frac{\ln\left(1+\frac{2\tan x}{1-\tan x}\right)}{1+\frac{2\tan x}{1-\tan x}}\frac{1+\frac{2\tan x}{1-\tan x}}{x}}=e^{\lim\limits_{x\to 0}\frac{\tan\left(\frac{\pi}{4}+x\right)}{x}}=e^{\lim\limits_{x\to 0}\frac{1}{\cos^2\left(\frac{\pi}{4}+x\right)}}=e^{2}$$

$5.proof:$

f有二阶导数说明f一阶导数可微连续

$$\lim_{h\to 0}\frac{\left[f(x+h)-f(x)\right]-\left[f(x)-f(x-h)\right]}{h^2}=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}$$

$$=\frac{1}{2}\lim_{h\to 0}\frac{f'(x+h)-f'(x)+f'(x)-f'(x-h)}{h}=f''(x)$$

11/14 homework

$1.proof$:

$Solution 1$:

$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 : C\delta_\varepsilon^\mu < \varepsilon, s.t. \forall x, y \in I : 0 < |x-y| < \delta_\varepsilon, s.t.$

$|f(x)-f(y)| < C|x-y|^\mu < C\delta_\varepsilon^\mu < \varepsilon$

因此，$f(x)$在I上一致连续.

假如I为无穷区间，不妨只考虑$I=[0,+\infty)$的情况，其他情况类似.

不妨设$f(0)=0$，否则用$f(x)-f(0)$代替$f(x)$

只需证：$f(0)\equiv 0$

$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \dfrac{C}{N^{\mu-1}} < \varepsilon, s.t. \forall n > N, \left|f\left(x+\dfrac{1}{n}\right)-f(x)\right| \le C\dfrac{1}{n^\mu}$

$then\ \ \forall k \in \mathbb{N} : k \le n,$

$$\left|f\left(\frac{k}{n}\right)\right| = \left|f\left(\frac{k}{n}\right)-f(0)\right| \le \sum_{i=0}^{k-1}\left|f\left(\frac{i+1}{n}\right)-f\left(\frac{i}{n}\right)\right| \le \sum_{i=0}^{n-1}\left|f\left(\frac{i+1}{n}\right)-f\left(\frac{i}{n}\right)\right| \le \frac{C}{n^{\mu-1}} < \varepsilon$$

由ε任意性：$f\left(\dfrac{k}{n}\right)\equiv 0$

$\forall \varepsilon > 0, for\ a\ given\ n : n > N \wedge n > \dfrac{1}{\delta_\varepsilon}$,对于$t\in[0,1], \exists k \in \mathbb{N} : k \le n, s.t.\ t \in \left[\dfrac{k-1}{n}, \dfrac{k}{n}\right], \left|t-\dfrac{k}{n}\right| < \delta_\varepsilon$

$then\ \ |f(t)| = \left|f(t)-f\left(\dfrac{k}{n}\right)\right| < \varepsilon$

由ε任意性：$f(t)\equiv 0, \forall t \in [0,1]$

since $f(1)=0, similarly, f(t)\equiv 0, \forall t\in[1,2], \ldots$

$by\ induction,, f(t)\equiv 0, \forall t \in [0,+\infty]$

若I为有限区间，不妨设$I=[0,1]$,由上述证明可知$f(t)\equiv 0, \forall t\in I$，

注：这里不妨设为闭区间是因为如果I为开区间我们可以不妨令$0\in I$，

将I从0处分开成正负两个在0处闭的区间，显然由题目给出的不等式，f在I上有界

所以不妨在$f(1)$处补充定义为$\lim\limits_{x\to 1^-} f(x)$.

2. 证: $f(x)g'(x) = f'(x)g(x)$. 设 $h(x) = \frac{f(x)}{g(x)}$. $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} = 0$

$\Rightarrow$ $h(x)$ 为常值函数. 记 $h(x) = c \Rightarrow f(x) = cg(x)$. $\forall x \in I$.

若 $g(x) = 0$. $f(x) = x$. 则 $\begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = 0$ 仍成立. 但 $f(x) \neq cg(x) = 0$.

3. 证: 不妨设 $0 \in I$. 否则平移 f. ~~则对于~~

~~则 $\frac{f(x)-f(0)}{x-0} = f'(\xi_x) = a$ 其中 ξ_x 介于 $0, x$ 之间 $\forall x \in I \setminus \{0\}$.~~

则. $\forall x \in I$. 有 $f(x) - f(0) = f'(\xi_x)(x-0)$ 其中 ξ_x 介于 $0, x$ 之间.

~~取 $a = f'$~~ 即 $f(x) = ax + f(0)$. 取 $b = f(0)$. 为常数 得证!

4. 证: 设 $F(x) = x^a f(x)$. $F'(x) = x^{a-1}[xf'(x) + af(x)] = 0$. 则 $F(x)$ 为常值函数

记 $F(x) = c \Rightarrow f(x) = cx^{-a}$. $\forall x \in I$

5. (1). 设 $F(x) = f(x) - \frac{b}{a}e^{ax}$. 则 $F'(x) = f'(x) - be^{ax} = 0$. 则 $F(x)$ 为常值函数

记 $F(x) = c \Rightarrow f(x) = \frac{b}{a}e^{ax} + c$. $\forall x \in I$

(2) 设 $F(x) = (f(x) + \frac{b}{a})e^{-ax}$. $F'(x) = [f'(x) - af(x) - b]e^{-ax}$. 则 $F(x)$ 为常值函数

$\Rightarrow$ 记 $F(x) = c \Rightarrow f(x) = ce^{ax} - \frac{b}{a}$

6. (2) $f(x) = \arctan x - \arcsin\frac{x}{\sqrt{1+x^2}}$. $\Rightarrow f'(x) = \frac{1}{1+x^2} - \frac{1}{\sqrt{1-\frac{x^2}{1+x^2}}}\left(\frac{1}{\sqrt{1+x^2}} - \frac{x^2}{(1+x^2)\sqrt{1+x^2}}\right)$

$= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0 \Rightarrow f(x)$ 为常值函数. 注意到 $f(0) = 0$

则 $f(x) = f(0) = 0$. $\Rightarrow \arctan x = \arcsin\frac{x}{\sqrt{1+x^2}}$

(4). $f(x) := 2\arctan x + \arcsin\frac{2x}{1+x^2}$ $(|x| \geq 1)$, $f'(x) = \frac{2}{1+x^2} + \frac{1}{\sqrt{1-(\frac{2x}{1+x^2})^2}} \cdot \frac{2(1+x^2) - 2x\cdot(2x)}{(1+x^2)^2}$

$= \frac{2}{1+x^2} + \frac{1+x^2}{x^2-1} \cdot \frac{2(1-x^2)}{(1+x^2)^2} = 0$. $|x| = 1$ 时. $f(x) = \pi\,\mathrm{sgn}\,x$

$\Rightarrow f(x)$ 为常值函数

显然 $f(x)$ 连续. 则. $f(x) = \pi\,\mathrm{sgn}\,x$. $(|x| \geq 1)$

已经不想打字了 hiahia~

1.(2). $y=\frac{2x}{1+x^2}$ $\quad y'=\frac{2(1+x^2)-4x^2}{(1+x^2)^2}=\frac{2(1-x^2)}{(1+x^2)^2}$

⇒ y在 $(-\infty,-1)$, $(1,+\infty)$ 上↓, $(-1,1)$ 上↑

1.(4) $y=x^n e^{-x}$ $(n\in N)$. $\quad y'=(nx^{n-1}-x^n)e^{-x}=x^{n-1}e^{-x}(n-x)$.

⇒ y在 ~~在~~ $(-\infty,0)$, $(n,+\infty)$ 上↓, $(0,n)$上↑

2.(2). LHS: 只需证: $f(x):=\ln(1+x)+\frac{1}{2}x^2-x>0$ $\quad (x>0)$

$f'(x)=\frac{1}{1+x}+x-1=\frac{x^2}{1+x}>0 \Rightarrow f(x)\uparrow \Rightarrow f(x)>f(0)=0$ 得证.

RHS: 只需证: $g(x):=\ln(1+x)-x<0$. $(x>0)$.

$g'(x)=\frac{1}{1+x}-1=-\frac{x}{1+x}<0 \Rightarrow g(x)\downarrow \Rightarrow g(x)<g(0)=0$. 得证!

2.(4). 只需证: $e^{2x}+\frac{x+1}{x-1}<0$ $(0<x<1)$. 只需证: $(x-1)e^{2x}+x+1>0$.

$f(x):=(x-1)e^{2x}+(x+1) \Rightarrow f'(x)=(2x-1)e^{2x}+1$, $f''(x)=4xe^{2x}>0 \Rightarrow f'(x)\uparrow$

$f'(x)>f'(0)=0 \Rightarrow f(x)\uparrow \Rightarrow f(x)>f(0)=0$ 得证.

2.(6). LHS: 只需证: $\arctan x<(x+1)\ln(1+x)$. $(0<x<1)$.

$f(x):=(x+1)\ln(1+x)-\arctan x$ $(0<x<1)$. $\quad f'(x)=1+\ln(1+x)-\frac{1}{1+x^2}$

$f''(x)=\frac{1}{1+x}+\frac{2x}{(1+x^2)^2}=\frac{(1+x^2)^2+2x(1+x)}{(1+x)(1+x^2)^2}>0 \Rightarrow f'(x)\uparrow$

$\Rightarrow f'(x)>f'(0)=0 \Rightarrow f(x)\uparrow \Rightarrow f(x)>f(0)=0$ 得证.

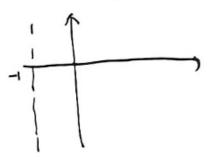
RHS: 只需证: $g(x):=\arctan x-\ln(1+x)$ $(0<x<1)$. $\quad g'(x)=\frac{1}{1+x^2}-\frac{1}{1+x}=\frac{x(1-x)}{(1+x)(1+x^2)}$

$\Rightarrow g(x)\uparrow \Rightarrow g(x)>g(0)=0$ 得证!

4. 证: $f(x):=\ln(1+x)-\arctan x-c$ $(x>-1)$. $\quad f'(x)=\frac{1}{1+x}-\frac{1}{1+x^2}=\frac{x(x-1)}{(1+x)(1+x^2)}$.

f(x)在 $(-1,0)$↑, $(0,1)$↓ $(1,+\infty)$↑

$\lim_{x\to(-1)^+}f(x)=-\infty$ $\quad \lim_{x\to+\infty}f(x)=+\infty$. $f(0)=-c$. $f(1)=\ln 2-\frac{\pi}{4}-c$.



显然. 得证!

7.(1). $f(x):=\sin x+\cos x+x^2-x-1$ $(x>0)$. $f'(x)=\cos x-\sin x+2x-1$. $f''(x)=-\sin x-\cos x+2>0$

$\Rightarrow f'(x)\uparrow \Rightarrow f'(x)>f'(0)=0 \Rightarrow f(x)\uparrow \Rightarrow f(x)>f(0)=0$.

7.(3). LHS: $f(x):=\cos x+\frac{1}{2}x^2-1$ $(x\neq 0)$. $f'(x)=-\sin x+x$ $f''(x)=1-\cos x\geq 0$.

由于f(x)为偶函数. 所以不妨设 $x>0$. $\Rightarrow f'(x)\uparrow \Rightarrow f'(x)>f'(0)=0. \Rightarrow f(x)\uparrow$

$f(x)>f(0)=0$. 得证!

RHS: $g(x):=\cos x-1+\frac{1}{2}x^2-\frac{1}{24}x^4$. $(x\neq 0)$. 不妨设 $x>0$.

$\Rightarrow g'(x)=-\sin x+x-\frac{1}{6}x^3 \Rightarrow g''(x)=-\cos x+1-\frac{1}{2}x^2<0$ (由LHS).

$\Rightarrow g'(x)\downarrow \Rightarrow g'(x)<g'(0)=0 \Rightarrow g(x)\downarrow \Rightarrow g(x)<g(0)=0$. 得证!

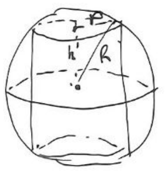
9.(1). $y=x^3+3x^2-9x+4$. $y'=3x^2+6x-9=3(x+3)(x-1)$.

极大值为 $y|_{x=-3}=-27+27+27+4=31$. 极小值为 $y|_{x=1}=-1$.

9.(4). $y=xe^{-x^2}$. $y'=(1-2x^2)e^{-x^2}$

极小值为 $y|_{x=-\frac{\sqrt{2}}{2}}=-\sqrt{\frac{1}{2e}}$. 极大值为 $y|_{x=\frac{\sqrt{2}}{2}}=\sqrt{\frac{1}{2e}}$

12.(1)不妨考虑. 圆柱的上下面边界都在球面上. 否则 存在更大体积的圆柱.



$h^2+r^2=R^2$. $V=2\pi r^2h$ ~~$=2\pi\sqrt{R^2-h^2}\cdot h\leq 2\pi\frac{R^2}{2}$~~

$\Rightarrow V=2\pi h(R^2-h^2)$. $\frac{dV}{dh}=2\pi(R^2-3h^2)$.

$\Rightarrow V_{max}=V|_{h=\frac{\sqrt{3}}{3}R}=\frac{4\sqrt{3}}{9}\pi R^3$.

13.(1).

$V=\pi r^2h+\frac{2}{3}\pi r^3$. 为定值.

$S=\pi r^2+2\pi rh+2\pi r^2=3\pi r^2+2\pi rh$

$\Rightarrow h=\frac{\frac{V}{\pi}-\frac{2}{3}r^3}{r^2} \Rightarrow S=3\pi r^2+2\pi r\cdot\frac{\frac{V}{\pi}-\frac{2}{3}r^3}{r^2}=3\pi r^2+\frac{2V}{r}-\frac{4}{3}\pi r^2=\frac{5}{3}\pi r^2+\frac{2V}{r}$

Let $\frac{dS}{dr}=\frac{10}{3}\pi r-\frac{2V}{r^2}=0 \Rightarrow r=\sqrt[3]{\frac{3V}{5\pi}}$

$\Rightarrow S_{min}=S|_{r=\sqrt[3]{\frac{3V}{5\pi}}}=3\pi\sqrt[3]{\frac{9V^2}{25\pi^2}}+2\pi\sqrt[3]{\frac{3V}{5\pi}}\cdot h$