11/6 homework

$$1.(2)\ y=\frac{x}{\sqrt[3]{1+x}}=x(1+x)^{-\frac{1}{3}}$$

$$y^{(n)}=\sum_{k=0}^{n}C_n^k x^{(k)}\left((1+x)^{-\frac{1}{3}}\right)^{(n-k)}=C_n^0 x\left((1+x)^{-\frac{1}{3}}\right)^{(n)}+C_n^1\left((1+x)^{-\frac{1}{3}}\right)^{(n-1)}$$

$$=x\left((1+x)^{-\frac{1}{3}}\right)^{(n)}+n\left((1+x)^{-\frac{1}{3}}\right)^{(n-1)}$$

$$\left((1+x)^{-\frac{1}{3}}\right)^{(n)}=(1+x)^{-\frac{1}{3}-n}\prod_{k=0}^{n-1}\left(-\frac{1}{3}-k\right)=(1+x)^{-\frac{1}{3}-n}(-1)^n\prod_{k=0}^{n-1}\left(\frac{1}{3}+k\right)=(1+x)^{-\frac{1}{3}-n}(-1)^n\frac{\Gamma\left(n+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\frac{1}{3}$$

$$y^{(n)}=x\left((1+x)^{-\frac{1}{3}}\right)^{(n)}+n\left((1+x)^{-\frac{1}{3}}\right)^{(n-1)}=x(1+x)^{-\frac{1}{3}-n}(-1)^n\prod_{k=0}^{n-1}\left(\frac{1}{3}+k\right)+n(1+x)^{\frac{2}{3}-n}(-1)^{n-1}\prod_{k=0}^{n-2}\left(\frac{1}{3}+k\right)$$

$$=x(1+x)^{-\frac{1}{3}-n}(-1)^n\prod_{k=0}^{n-1}\left(\frac{1}{3}+k\right)-n(1+x)(1+x)^{-\frac{1}{3}-n}(-1)^n\prod_{k=0}^{n-2}\left(\frac{1}{3}+k\right)$$

$$=\left(n-\frac{2}{3}\right)x(1+x)^{-\frac{1}{3}-n}(-1)^n\prod_{k=0}^{n-2}\left(\frac{1}{3}+k\right)-n(1+x)(1+x)^{-\frac{1}{3}-n}(-1)^n\prod_{k=0}^{n-2}\left(\frac{1}{3}+k\right)$$

$$=\left[\left(n-\frac{2}{3}\right)x-n(1+x)\right](1+x)^{-\frac{1}{3}-n}(-1)^n\prod_{k=0}^{n-2}\left(\frac{1}{3}+k\right)$$

$$=\left(-\frac{2}{3}x-n\right)(1+x)^{-\frac{1}{3}-n}(-1)^n\prod_{k=0}^{n-2}\left(\frac{1}{3}+k\right)$$

$$1.(4)\ y=\frac{1}{x(1-x)}=\frac{1}{x}+\frac{1}{1-x}$$

$$y^{(n)}=\left(\frac{1}{x}+\frac{1}{1-x}\right)^{(n)}=\left(\frac{1}{x}\right)^{(n)}+\left(\frac{1}{1-x}\right)^{(n)}=\frac{(-1)^n n!}{x^{n+1}}+\frac{n!}{(1-x)^{n+1}}$$

$$1.(6)\ y=\frac{x}{\left(x^2-1\right)^2}=\frac{x}{x^2-1}\frac{1}{x^2-1}=\frac{1}{4}\left(\frac{1}{x-1}+\frac{1}{x+1}\right)\left(\frac{1}{x-1}-\frac{1}{x+1}\right)=\frac{1}{4}\left(\frac{1}{(x-1)^2}-\frac{1}{(x+1)^2}\right)$$

$$y^{(n)}=\frac{1}{4}\left(\frac{1}{(x-1)^2}\right)^{(n)}-\frac{1}{4}\left(\frac{1}{(x+1)^2}\right)^{(n)}=\frac{1}{4}\frac{(-1)^n(n+1)!}{(x-1)^{n+2}}-\frac{1}{4}\frac{(-1)^n(n+1)!}{(x+1)^{n+2}}$$

$$1.(8)\ y=\sin^6 x+\cos^6 x=\left(\sin^2 x+\cos^2 x\right)\left(\sin^4 x-\sin^2 x\cos^2 x+\cos^4 x\right)$$

$$=\sin^4 x-\sin^2 x\cos^2 x+\cos^4 x=\left(\sin^2 x+\cos^2 x\right)^2-3\sin^2 x\cos^2 x=1-3\sin^2 x\cos^2 x$$

$$=1-\frac{3}{4}\sin^2(2x)=1-\frac{3}{4}\frac{1-\cos(4x)}{2}=\frac{5}{8}+\frac{3}{8}\cos(4x)$$

$$y^{(n)}=\left(\frac{5}{8}+\frac{3}{8}\cos(4x)\right)^{(n)}=\frac{3}{8}\left(\cos(4x)\right)^{(n)}=\frac{3}{8}\cdot 4^n\cos\left(4x+\frac{n\pi}{2}\right)=3\cdot 2^{2n-3}\cdot\cos\left(4x+\frac{n\pi}{2}\right)$$

$2.proof:$

$Solution1:$

$$\left[e^{ax}\sin(bx+c)\right]^{(n)}=\operatorname{Im}\left\{\left[e^{ax}\cdot e^{i(bx+c)}\right]^{(n)}\right\}=\operatorname{Im}\left\{\left[e^{ax+i(bx+c)}\right]^{(n)}\right\}$$

$$=\operatorname{Im}\left\{(a+ib)^n e^{ax+i(bx+c)}\right\}=\operatorname{Im}\left\{\left(\sqrt{a^2+b^2}\cdot e^{i\arctan\left(\frac{b}{a}\right)}\right)^n e^{ax+i(bx+c)}\right\}$$

$$=\left(a^2+b^2\right)^{\frac{n}{2}}\operatorname{Im}\left\{e^{ax+i\left(bx+c+n\arctan\left(\frac{b}{a}\right)\right)}\right\}=\left(a^2+b^2\right)^{\frac{n}{2}}e^{ax}\sin\left(bx+c+n\arctan\left(\frac{b}{a}\right)\right)$$

$$\left[e^{ax}\cos(bx+c)\right]^{(n)}=\left[e^{ax}\cos(-bx-c)\right]^{(n)}=\left[e^{ax}\sin\left(\frac{\pi}{2}+bx+c\right)\right]^{(n)}$$

$$\overset{c'\triangleq c+\frac{\pi}{2}}{=}\left[e^{ax}\sin(bx+c')\right]^{(n)}=\left(a^2+b^2\right)^{\frac{n}{2}}e^{ax}\sin\left(bx+c'+n\arctan\left(\frac{b}{a}\right)\right)$$

$$=\left(a^2+b^2\right)^{\frac{n}{2}}e^{ax}\sin\left(bx+c+\frac{\pi}{2}+n\arctan\left(\frac{b}{a}\right)\right)=\left(a^2+b^2\right)^{\frac{n}{2}}e^{ax}\cos\left(bx+c+n\arctan\left(\frac{b}{a}\right)\right)$$

$Solution2:$

$by\ induction:$

$$assume\ that:\left[e^{ax}\sin(bx+c)\right]^{(n-1)}=\left(a^2+b^2\right)^{\frac{n-1}{2}}e^{ax}\sin\left(bx+c+(n-1)\arctan\left(\frac{b}{a}\right)\right),\forall n\in\mathbb{N}$$

$$\left[e^{ax}\sin(bx+c)\right]^{(n)}=\left\{\left[e^{ax}\sin(bx+c)\right]^{(n-1)}\right\}'=\left[\left(a^2+b^2\right)^{\frac{n-1}{2}}e^{ax}\sin\left(bx+c+(n-1)\arctan\left(\frac{b}{a}\right)\right)\right]'$$

$$=\left(a^2+b^2\right)^{\frac{n-1}{2}}e^{ax}\left[\sin\left(bx+c+(n-1)\arctan\left(\frac{b}{a}\right)\right)\right]'+\left(a^2+b^2\right)^{\frac{n-1}{2}}\left(e^{ax}\right)'\sin\left(bx+c+(n-1)\arctan\left(\frac{b}{a}\right)\right)$$

$$=\left(a^2+b^2\right)^{\frac{n-1}{2}}be^{ax}\cos\left(bx+c+(n-1)\arctan\left(\frac{b}{a}\right)\right)+\left(a^2+b^2\right)^{\frac{n-1}{2}}ae^{ax}\sin\left(bx+c+(n-1)\arctan\left(\frac{b}{a}\right)\right)$$

$$=\left(a^2+b^2\right)^{\frac{n-1}{2}}e^{ax}\left[b\cos\left(bx+c+(n-1)\arctan\left(\frac{b}{a}\right)\right)+a\sin\left(bx+c+(n-1)\arctan\left(\frac{b}{a}\right)\right)\right]$$

$$=\left(a^2+b^2\right)^{\frac{n-1}{2}}e^{ax}\left(a^2+b^2\right)^{\frac{1}{2}}\sin\left(bx+c+n\arctan\left(\frac{b}{a}\right)\right)=\left(a^2+b^2\right)^{\frac{n}{2}}e^{ax}\sin\left(bx+c+n\arctan\left(\frac{b}{a}\right)\right)$$

$$\Rightarrow\left[e^{ax}\sin(bx+c)\right]^{(n)}=\left(a^2+b^2\right)^{\frac{n}{2}}e^{ax}\sin\left(bx+c+n\arctan\left(\frac{b}{a}\right)\right)$$

$$similarly,\left[e^{ax}\cos(bx+c)\right]^{(n)}=\left(a^2+b^2\right)^{\frac{n}{2}}e^{ax}\cos\left(bx+c+n\arctan\left(\frac{b}{a}\right)\right)$$

$7.(1)\ y=x^3e^{2x}$

$$y^{(n)}=\sum_{k=0}^{n}\left(x^3\right)^{(k)}\left(e^{2x}\right)^{(n-k)}=x^3\left(e^{2x}\right)^{(n)}+\left(x^3\right)^{(1)}\left(e^{2x}\right)^{(n-1)}+\left(x^3\right)^{(2)}\left(e^{2x}\right)^{(n-2)}+\left(x^3\right)^{(3)}\left(e^{2x}\right)^{(n-3)}$$

$$=x^3 2^n e^{2x}+3x^2\cdot 2^{n-1}e^{2x}+6x\cdot 2^{n-2}e^{2x}+6\cdot 2^{n-3}e^{2x}=e^{2x}\cdot 2^{n-2}\left(4x^3+6x^2+6x+3\right)$$

$7.(2)\ y=x^2\ln x$

$$y^{(n)}=\sum_{k=0}^{n}\left(x^2\right)^{(k)}\left(\ln x\right)^{(n-k)}=\left(x^2\right)^{(0)}\left(\ln x\right)^{(n)}+\left(x^2\right)^{(1)}\left(\ln x\right)^{(n-1)}+\left(x^2\right)^{(2)}\left(\ln x\right)^{(n-2)}$$

$$=x^2\left(\ln x\right)^{(n)}+2x\left(\ln x\right)^{(n-1)}+2\left(\ln x\right)^{(n-2)}$$

$$=x^2\frac{\left(-1\right)^{n-1}\left(n-1\right)!}{x^n}+2x\frac{\left(-1\right)^{n-2}\left(n-2\right)!}{x^{n-1}}+2\frac{\left(-1\right)^{n-3}\left(n-3\right)!}{x^{n-2}}$$

$$=\frac{\left(-1\right)^{n-1}\left(n-1\right)!}{x^{n-2}}+2\frac{\left(-1\right)^{n-2}\left(n-2\right)!}{x^{n-2}}+2\frac{\left(-1\right)^{n-3}\left(n-3\right)!}{x^{n-2}}$$

$$=\frac{\left(-1\right)^{n-1}\left(n-3\right)!}{x^{n-2}}\left[\left(n-1\right)\left(n-2\right)-2\left(n-2\right)+2\right]$$

$$=\frac{\left(-1\right)^{n-1}\left(n-3\right)!}{x^{n-2}}\left(n^2-5n+8\right)$$

$7.(4)\ y=\cosh ax\sin bx=\dfrac{e^x+e^{-x}}{2}\operatorname{Im}\left(e^{ibx}\right)=\operatorname{Im}\left(e^{ibx}\dfrac{e^x+e^{-x}}{2}\right)=\dfrac{1}{2}\operatorname{Im}\left(e^{(1+ib)x}+e^{(-1+ib)x}\right)$

$$y^{(n)}=\frac{1}{2}\operatorname{Im}\left(\left(e^{(1+ib)x}\right)^{(n)}+\left(e^{(-1+ib)x}\right)^{(n)}\right)=\frac{1}{2}\operatorname{Im}\left(\left(1+ib\right)^n e^{(1+ib)x}+\left(-1+ib\right)^n e^{(-1+ib)x}\right)$$

$$=\frac{1}{2}\operatorname{Im}\left(\left(\sqrt{1+b^2}e^{\arctan b}\right)^n e^{(1+ib)x}+\left(\sqrt{1+b^2}e^{-\arctan b}\right)^n e^{(-1+ib)x}\right)$$

$$=\frac{\left(1+b^2\right)^{\frac{n}{2}}}{2}\left(e^{x+n\arctan b}+e^{-x-n\arctan b}\right)\operatorname{Im}\left(e^{ibx}\right)=\frac{\left(1+b^2\right)^{\frac{n}{2}}}{2}\left(e^{x+n\arctan b}+e^{-x-n\arctan b}\right)\sin bx$$

$$=\left(1+b^2\right)^{\frac{n}{2}}\cosh\left(x+n\arctan b\right)\sin bx$$

$7.(6)\ y=\ln x\sin x$

考虑: $y=e^{ix}\ln x,\operatorname{Im}\left(y\right)=\ln x\sin x,\operatorname{Im}\left(y^{(n)}\right)=\left(\ln x\sin x\right)^{(n)}$

$$e^{-ix}y=\ln x$$

$$\frac{\left(-1\right)^{n-1}\left(n-1\right)!}{x^n}=\left(e^{-ix}y\right)^{(n)}=\sum_{k=0}^{n}\left(e^{-ix}\right)^k y^{(n-k)}=e^{-ix}\sum_{k=0}^{n}\left(-i\right)^k y^{(n-k)}$$

$$\frac{\left(-1\right)^n n!}{x^{n+1}}=e^{-ix}\sum_{k=0}^{n+1}\left(-i\right)^k y^{(n+1-k)}=e^{-ix}\sum_{k=-1}^{n}\left(-i\right)^{k+1}y^{(n-k)}=e^{-ix}\left(-i\right)\sum_{k=-1}^{n}\left(-i\right)^k y^{(n-k)}$$

$$=e^{-ix}\left(-i\right)\left(\sum_{k=0}^{n}\left(-i\right)^k y^{(n-k)}+iy^{(n+1)}\right)=\left(-i\right)e^{-ix}\sum_{k=0}^{n}\left(-i\right)^k y^{(n-k)}+\left(-i\right)e^{-ix}iy^{(n+1)}$$

$$=\left(-i\right)e^{-ix}\sum_{k=0}^{n}\left(-i\right)^k y^{(n-k)}+e^{-ix}y^{(n+1)}=\left(-i\right)\frac{\left(-1\right)^{n-1}\left(n-1\right)!}{x^n}+e^{-ix}y^{(n+1)}$$

$$\Rightarrow y^{(n+1)}=\left(\frac{\left(-1\right)^n n!}{x^{n+1}}+i\frac{\left(-1\right)^{n-1}\left(n-1\right)!}{x^n}\right)e^{ix}=\left(\frac{\left(-1\right)^n n!}{x^{n+1}}+i\frac{\left(-1\right)^{n-1}\left(n-1\right)!}{x^n}\right)\left(\cos x+i\sin x\right)$$

$$=\left(\frac{\left(-1\right)^n n!}{x^{n+1}}\cos x+\frac{\left(-1\right)^n\left(n-1\right)!}{x^n}\sin x\right)+i\left(\frac{\left(-1\right)^n n!}{x^{n+1}}\sin x+\frac{\left(-1\right)^{n-1}\left(n-1\right)!}{x^n}\cos x\right)$$

$$\Rightarrow\left(\ln x\sin x\right)^{(n)}=\operatorname{Im}\left(y^{(n)}\right)=\frac{\left(-1\right)^{n-1}n!}{x^n}\sin x+\frac{\left(-1\right)^n\left(n-1\right)!}{x^{n-1}}\cos x$$

$$11.(1)\ y-\frac{1}{2}\sin y=x$$

$$\Rightarrow y'-\frac{1}{2}y'\cos y=1\Rightarrow y'=\frac{2}{2-\cos y}$$

$$\Rightarrow y''-\frac{1}{2}y''\cos y+\frac{1}{2}y'^2\sin y=0\Rightarrow y''=-\frac{y'^2\sin y}{2-\cos y}=-\frac{4\sin y}{(2-\cos y)^3}$$

$$\Rightarrow y'''-\frac{1}{2}y'''\cos y+\frac{1}{2}y''y'\sin y+y'y''\sin y+\frac{1}{2}y'^3\cos y=0$$

$$\Rightarrow y'''=-\frac{\frac{1}{2}\left[-\frac{4\sin y}{(2-\cos y)^3}\right]\frac{2}{2-\cos y}\sin y+\frac{2}{2-\cos y}\left[-\frac{4\sin y}{(2-\cos y)^3}\right]\sin y+\frac{1}{2}\left(\frac{2}{2-\cos y}\right)^3\cos y}{1-\frac{1}{2}\cos y}$$

$$=\frac{\frac{4\sin y}{(2-\cos y)^3}\frac{1}{2-\cos y}\sin y-\frac{2}{2-\cos y}\frac{4\sin y}{(2-\cos y)^3}\sin y+\frac{1}{2}\frac{8}{(2-\cos y)^3}\cos y}{1-\frac{1}{2}\cos y}$$

$$=\frac{-\frac{4\sin^2 y}{(2-\cos y)^4}+\frac{4\cos y}{(2-\cos y)^3}}{1-\frac{1}{2}\cos y}=\frac{4\frac{(2-\cos y)\cos y-\sin^2 y}{(2-\cos y)^4}}{1-\frac{1}{2}\cos y}=\frac{16\cos y-8}{(2-\cos y)^5}$$

$$11.(2)\ y^2+2\ln y=x$$

$$\Rightarrow 2y'y+\frac{2}{y}=1\Rightarrow y'=\frac{1-\frac{2}{y}}{2y}=\frac{y-2}{2y^2}$$

$$y'=\frac{1}{2}\frac{1}{y^2}(y-2)\Rightarrow y''=\frac{1}{2}\left[\frac{1}{y^2}(y-2)\right]'=\frac{1}{2}\left[-\frac{2y'}{y^3}(y-2)+\frac{y'}{y^2}\right]$$

$$=\frac{1}{2}\left[\frac{yy'-2y'(y-2)}{y^3}\right]=\frac{4y'-yy'}{2y^3}=\frac{4-y}{2y^3}\frac{y-2}{2y^2}=\frac{-y^2+6y-8}{4y^5}=\frac{1}{4y^5}\left(-y^2+6y-8\right)$$

$$\Rightarrow y'''=\left[\frac{1}{4y^5}\left(-y^2+6y-8\right)\right]'=\frac{1}{4}\left[\frac{1}{y^5}\left(-y^2+6y-8\right)\right]'$$

$$=\frac{1}{4}\left[-\frac{5y'}{y^6}\left(-y^2+6y-8\right)+\frac{1}{y^5}(-2y+6)y'\right]=\frac{1}{4}\frac{5y^2-30y+40-2y^2+6}{y^6}y'$$

$$=\frac{-y^2+30y+46}{4y^6}\frac{y-2}{2y^2}=\frac{-y^3+32y^2-14y-92}{8y^8}$$

$$11.(3)\, x^3+y^3-3axy=0$$

$$\Rightarrow 3x^2+3y^2y'-3ay-3axy'=0 \Rightarrow x^2+y^2y'-ay-axy'=0$$

$$\Rightarrow y'=\frac{x^2-ay}{ax-y^2}$$

$$\Rightarrow 2x+2yy'^2+y^2y''-ay'-ay'-axy''=0 \Rightarrow 2x+2yy'^2+y^2y''-2ay'-axy''=0$$

$$\Rightarrow y''=\frac{2x+2yy'^2-2ay'}{ax-y^2}=\frac{2x+2y\left(\frac{x^2-ay}{ax-y^2}\right)^2-2a\frac{x^2-ay}{ax-y^2}}{ax-y^2}$$

$$=\frac{2x\left(ax-y^2\right)^2+2y\left(x^2-ay\right)^2-2a\left(x^2-ay\right)\left(ax-y^2\right)}{\left(ax-y^2\right)^3}$$

$$=\frac{\left(2xy^4-4ax^2y^2+2a^2x^3\right)+\left(2x^4y-4ax^2y^2+2a^2y^3\right)-\left(2a^2x^3-2ax^2y^2-2a^3xy+2a^2y^3\right)}{\left(ax-y^2\right)^3}$$

$$=\frac{2xy^4+2x^4y-6ax^2y^2+2a^3xy}{\left(ax-y^2\right)^3}$$

$$2x+2yy'^2+y^2y''-2ay'-axy''=0$$

$$\Rightarrow 2+2y'^3+4yy'y''+y^2y'''-3ay''-axy'''=0$$

$$\Rightarrow y'''=\frac{2+2y'^3+4yy'y''-3ay''}{ax-y^2}$$

$$=\frac{2+2\left(\frac{x^2-ay}{ax-y^2}\right)^3+4y\frac{x^2-ay}{ax-y^2}\frac{2xy^4+2x^4y-6ax^2y^2+2a^3xy}{\left(ax-y^2\right)^3}-3a\frac{2xy^4+2x^4y-6ax^2y^2+2a^3xy}{\left(ax-y^2\right)^3}}{ax-y^2}$$

$$=\frac{2\left(ax-y^2\right)^4+2\left(ax-y^2\right)\left(x^2-ay\right)^3+4y\left(x^2-ay\right)\left(2xy^4+2x^4y-6ax^2y^2+2a^3xy\right)}{\left(ax-y^2\right)^5}$$

$$-\frac{3a\left(ax-y^2\right)\left(2xy^4+2x^4y-6ax^2y^2+2a^3xy\right)}{\left(ax-y^2\right)^5}$$

$$12.(1)\ y=x+\ln x\Rightarrow 1=x'+\frac{x'}{x}\Rightarrow x'=\frac{1}{1+\frac{1}{x}}=\frac{x}{x+1}$$

$$x''=\left(\frac{x}{x+1}\right)'=\left(1-\frac{1}{x+1}\right)'=\frac{x'}{(x+1)^2}=\frac{\frac{x}{x+1}}{(x+1)^2}=\frac{x}{(x+1)^3}$$

$$x'''=\left(\frac{x}{(x+1)^3}\right)'=\frac{x'}{(x+1)^3}+x\left(\frac{1}{(x+1)^3}\right)'=\frac{x'}{(x+1)^3}-\frac{3xx'}{(x+1)^4}=\frac{x'(x+1)-3xx'}{(x+1)^4}=\frac{x'-2xx'}{(x+1)^4}$$

$$=\frac{\frac{x}{x+1}-2x\frac{x}{x+1}}{(x+1)^4}=\frac{x-2x^2}{(x+1)^5}$$

$$12.(2)\ y=x+e^x\Rightarrow 1=x'+x'e^x\Rightarrow x'=\frac{1}{1+e^x}$$

$$\Rightarrow x''=\left(\frac{1}{1+e^x}\right)'=\frac{x'e^x}{(1+e^x)^2}=\frac{e^x}{(1+e^x)^3}$$

$$\Rightarrow x'''=\left(\frac{e^x}{(1+e^x)^3}\right)'=\frac{e^x}{(1+e^x)^3}x'+e^x\frac{-3}{(1+e^x)^4}e^x x'=\left(\frac{e^x}{(1+e^x)^3}-\frac{3e^{2x}}{(1+e^x)^4}\right)x'$$

$$=\left(\frac{e^x}{(1+e^x)^3}-\frac{3e^{2x}}{(1+e^x)^4}\right)\frac{1}{1+e^x}=\frac{e^x}{(1+e^x)^4}-\frac{3e^{2x}}{(1+e^x)^5}=\frac{e^x(1+e^x)-3e^{2x}}{(1+e^x)^5}=\frac{e^x-2e^{2x}}{(1+e^x)^5}$$

13.(1) $\begin{cases} x = 2t - t^2 \\ y = 3t - t^3 \end{cases}, \frac{dx}{dt} = 2 - 2t, \frac{dy}{dt} = 3 - 3t^2,$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{3-3t^2}{2-2t} = \frac{3}{2}(1+t)$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d\frac{3}{2}(1+t)}{dx} = \frac{3}{2}\frac{d(1+t)}{dx} = \frac{3}{2}\frac{dt}{dx} = \frac{3}{2}\frac{1}{2-2t} = \frac{3}{4(1-t)}$$

$$\frac{d^3y}{dx^3} = \frac{d\left(\frac{d^2y}{dx^2}\right)}{dx} = \frac{d\left(\frac{3}{4(1-t)}\right)}{dx} = \frac{3}{4}\frac{d\left(\frac{1}{1-t}\right)}{dx} = \frac{3}{4}\frac{d\left(\frac{1}{1-t}\right)}{dt}\frac{dt}{dx} = \frac{3}{4}\frac{1}{(1-t)^2}\frac{1}{2-2t} = \frac{3}{8(1-t)^3}$$

13.(2) $\begin{cases} x = a\cos^3 t \\ y = at\sin^3 t \end{cases}, \frac{dx}{dt} = -3a\sin t\cos^2 t, \frac{dy}{dt} = a\sin^3 t + 3at\sin^2 t\cos t$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{a\sin^3 t + 3at\sin^2 t\cos t}{-3a\sin t\cos^2 t} = -\frac{\sin^2 t + 3t\sin t\cos t}{3\cos^2 t}$$

$$\frac{d^2y}{dx^2} = \frac{d\left(-\frac{\sin^2 t + 3t\sin t\cos t}{3\cos^2 t}\right)}{dx} = \frac{d\left(-\frac{\sin^2 t + 3t\sin t\cos t}{3\cos^2 t}\right)}{dt}\frac{dt}{dx}$$

$$= -\frac{\cos^2 t\left(2\sin t\cos t + 3\sin t\cos t + 3t\cos^2 t - 3t\sin^2 t\right) - \left(\sin^2 t + 3t\sin t\cos t\right)\left(-2\sin t\cos t\right)}{3\cos^4 t}\frac{dt}{dx}$$

$$= -\frac{2\sin t\cos^3 t + 3\sin t\cos^3 t + 3t\cos^4 t - 3t\sin^2 t\cos^2 t + 2\sin^3 t\cos t + 6t\sin^2 t\cos^2 t}{3\cos^4 t}\frac{dt}{dx}$$

$$= -\frac{2\sin t\cos t + 3\sin t\cos^3 t + 3t\cos^2 t}{3\cos^4 t}\frac{dt}{dx} = -\frac{2\sin t + 3\sin t\cos^2 t + 3t\cos t}{3\cos^3 t}\frac{dt}{dx}$$

$$= -\frac{2\sin t + 3\sin t\cos^2 t + 3t\cos t}{3\cos^3 t}\frac{1}{-3a\sin t\cos^2 t} = \frac{2\sin t + 3\sin t\cos^2 t + 3t\cos t}{9a\sin t\cos^5 t}$$

$$\frac{d^3y}{dx^3} = \frac{d\left(\frac{d^2y}{dx^2}\right)}{dx} = \frac{d\left(\frac{d^2y}{dx^2}\right)}{dt}\frac{dt}{dx} = \frac{d\left(\frac{2\sin t + 3\sin t\cos^2 t + 3t\cos t}{9a\sin t\cos^5 t}\right)}{dt}\frac{dt}{dx}$$

$$= \frac{\sin t\cos^5 t\left(2\cos t + 3\cos^3 t - 6\sin^2 t\cos t + 3\cos t - 3t\sin t\right) - \left(2\sin t + 3\sin t\cos^2 t + 3t\cos t\right)\left(\cos^6 t - 5\sin^2 t\cos^4 t\right)}{9a\left(\sin t\cos^5 t\right)^2}\frac{dt}{dx}$$

$$= \frac{\sin t\cos^5 t\left(5\cos t + 3\cos^3 t - 6\sin^2 t\cos t - 3t\sin t\right) - \left(2\sin t + 3\sin t\cos^2 t + 3t\cos t\right)\left(\cos^6 t - 5\sin^2 t\cos^4 t\right)}{9a\left(\sin t\cos^5 t\right)^2}\frac{dt}{dx}$$

$$= \frac{5\sin t\cos^6 t + 3\sin t\cos^8 t - 6\sin^3 t\cos^6 t - 3t\sin^2 t\cos^5 t}{9a\left(\sin t\cos^5 t\right)^2}\frac{dt}{dx}$$

$$-\frac{2\sin t\cos^6 t + 3\sin t\cos^8 t + 3t\cos^7 t}{9a\left(\sin t\cos^5 t\right)^2}\frac{dt}{dx} + \frac{10\sin^3 t\cos^4 t + 15\sin^3 t\cos^6 t + 15t\sin^2 t\cos^5 t}{9a\left(\sin t\cos^5 t\right)^2}\frac{dt}{dx}$$

$$= \frac{1}{9a\left(\sin t\cos^5 t\right)^2}\left(3\sin t\cos^6 t - 3t\cos^7 t + 10\sin^3 t\cos^4 t + 9\sin^3 t\cos^6 t + 12t\sin^2 t\cos^5 t\right)\frac{dt}{dx}$$

$$= \frac{1}{9a\left(\sin t\cos^5 t\right)^2}\left(3\sin t\cos^6 t - 3t\cos^7 t + 10\sin^3 t\cos^4 t + 9\sin^3 t\cos^6 t + 12t\sin^2 t\cos^5 t\right)\frac{1}{-3a\sin t\cos^2 t}$$

$$= -\frac{3\sin t\cos^2 t - 3t\cos^3 t + 10\sin^3 t + 9\sin^3 t\cos^2 t + 12t\sin^2 t\cos t}{27a^2\sin^3 t\cos^8 t}$$

$14.(1)\ y=\left(x^3+2x\right)\sin^2 x$

$$\frac{dy}{dx}=\frac{d\left(\left(x^3+2x\right)\sin^2 x\right)}{dx}=\frac{d\left(x^3+2x\right)}{dx}\sin^2 x+\left(x^3+2x\right)\frac{d\left(\sin^2 x\right)}{dx}$$

$$=\left(3x^2+2\right)\sin^2 x+\left(x^3+2x\right)\frac{d\left(\sin^2 x\right)}{d\sin x}\frac{d\sin x}{dx}$$

$$=\left(3x^2+2\right)\sin^2 x+\left(x^3+2x\right)\sin 2x$$

$$=\left(3x^2+2\right)\frac{1-\cos 2x}{2}+\left(x^3+2x\right)\sin 2x$$

$$=\frac{3}{2}x^2+1-\left(\frac{3}{2}x^2+1\right)\cos 2x+\left(x^3+2x\right)\sin 2x$$

$$\frac{d^2y}{dx^2}=\frac{d\left(\frac{dy}{dx}\right)}{dx}=\frac{d\left(\frac{3}{2}x^2+1-\left(\frac{3}{2}x^2+1\right)\cos 2x+\left(x^3+2x\right)\sin 2x\right)}{dx}$$

$$=3x-\left(3x\right)\cos 2x+\left(3x^2+2\right)\sin 2x+\left(3x^2+2\right)\sin 2x+\left(2x^3+4x\right)\cos 2x$$

$$=3x+\left(6x^2+4\right)\sin 2x+\left(2x^3+x\right)\cos 2x$$

$$\frac{d^3y}{dx^3}=\frac{d\left(\frac{d^2y}{dx^2}\right)}{dx}=\frac{d\left(3x+\left(6x^2+4\right)\sin 2x+\left(2x^3+x\right)\cos 2x\right)}{dx}$$

$$=3+\left(12x\right)\sin 2x+\left(6x^2+1\right)\cos 2x+\left(12x^2+8\right)\cos 2x-\left(4x^3+2x\right)\sin 2x$$

$$=3+\left(-4x^3+10x\right)\sin 2x+\left(18x^2+9\right)\cos 2x$$

$$\Rightarrow\begin{cases}d^2y=\left[3x+\left(6x^2+4\right)\sin 2x+\left(2x^3+x\right)\cos 2x\right]dx^2\\ d^3y=\left[3+\left(-4x^3+10x\right)\sin 2x+\left(18x^2+9\right)\cos 2x\right]dx^3\end{cases}$$

$14.(2)\ y=e^{2x}\ln x$

$$\frac{dy}{dx}=\frac{d\left(e^{2x}\ln x\right)}{dx}=2e^{2x}\ln x+\frac{e^{2x}}{x}$$

$$\frac{d^2y}{dx^2}=\frac{d\left(\frac{dy}{dx}\right)}{dx}=\frac{d\left(2e^{2x}\ln x+\frac{e^{2x}}{x}\right)}{dx}=4e^{2x}\ln x+\frac{2e^{2x}}{x}+\frac{2e^{2x}}{x}-\frac{e^{2x}}{x^2}=4e^{2x}\ln x+\frac{4e^{2x}}{x}-\frac{e^{2x}}{x^2}$$

$$\frac{d^3y}{dx^3}=\frac{d\left(\frac{d^2y}{dx^2}\right)}{dx}=\frac{d\left(4e^{2x}\ln x+\frac{4e^{2x}}{x}-\frac{e^{2x}}{x^2}\right)}{dx}=8e^{2x}\ln x+\frac{4e^{2x}}{x}+\frac{8e^{2x}}{x}-\frac{4e^{2x}}{x^2}-\frac{2e^{2x}}{x^2}+\frac{2e^{2x}}{x^3}$$

$$=8e^{2x}\ln x+\frac{12e^{2x}}{x}-\frac{6e^{2x}}{x^2}+\frac{2e^{2x}}{x^3}$$

11/7 homework

$$1.(2)\vec{r}=a\cos\theta\vec{i}+b\sin\theta\vec{j}$$

$$\vec{r}'=\left(a\cos\theta\vec{i}+b\sin\theta\vec{j}\right)'=-a\sin\theta\vec{i}+b\cos\theta\vec{j}$$

$$1.(4)\vec{r}=ae^{b\theta}\cos\theta\vec{i}+ae^{b\theta}\sin\theta\vec{j}$$

$$\vec{r}'=\left(ae^{b\theta}\cos\theta\vec{i}+ae^{b\theta}\sin\theta\vec{j}\right)'=ae^{b\theta}\left(b\cos\theta-\sin\theta\right)\vec{i}+ae^{b\theta}\left(b\sin\theta+\cos\theta\right)\vec{j}$$

$$1.(7)\ \vec{r}=\left(R+r\cos\theta\right)\cos a\theta\vec{i}+\left(R+r\cos\theta\right)\sin a\theta\vec{j}+r\sin\theta\vec{k}$$

$$\vec{r}'=\left[\left(R+r\cos\theta\right)\cos a\theta\vec{i}+\left(R+r\cos\theta\right)\sin a\theta\vec{j}+r\sin\theta\vec{k}\right]'$$

$$=\left[-a\left(R+r\cos\theta\right)\sin a\theta-r\sin\theta\cos a\theta\right]\vec{i}+\left[a\left(R+r\cos\theta\right)\sin a\theta-r\sin\theta\sin a\theta\right]\vec{j}+r\cos\theta\vec{k}$$

$2.(1)\ proof:$

$$\frac{d\left(\vec{r}_1(t),\vec{r}_2(t),\vec{r}_3(t)\right)}{dt}=\frac{d\left(\vec{r}_1(t)\cdot\left(\vec{r}_2(t)\times\vec{r}_3(t)\right)\right)}{dt}$$

$$lemma\ 1:\left(\vec{a}(t_0)\cdot\vec{b}(t_0)\right)'=\vec{a}(t_0)\cdot\vec{b}'(t_0)+\vec{a}'(t_0)\cdot\vec{b}(t_0)$$

$$proof:\left(\vec{a}(t_0)\cdot\vec{b}(t_0)\right)'=\lim_{t\to t_0}\frac{\vec{a}(t)\cdot\vec{b}(t)-\vec{a}(t_0)\cdot\vec{b}(t_0)}{t-t_0}$$

$$=\lim_{t\to t_0}\frac{\left[\vec{a}(t)\cdot\vec{b}(t)-\vec{a}(t)\cdot\vec{b}(t_0)\right]+\left[\vec{a}(t)\cdot\vec{b}(t_0)-\vec{a}(t_0)\cdot\vec{b}(t_0)\right]}{t-t_0}$$

$$=\lim_{t\to t_0}\vec{a}(t)\cdot\frac{\vec{b}(t)-\vec{b}(t_0)}{t-t_0}+\lim_{t\to t_0}\frac{\vec{a}(t)\cdot\vec{b}(t_0)-\vec{a}(t_0)\cdot\vec{b}(t_0)}{t-t_0}$$

$$=\lim_{t\to t_0}\vec{a}(t)\cdot\lim_{t\to t_0}\frac{\vec{b}(t)-\vec{b}(t_0)}{t-t_0}+\lim_{t\to t_0}\frac{\vec{a}(t)-\vec{a}(t_0)}{t-t_0}\cdot\vec{b}(t_0)$$

$$=\vec{a}(t_0)\cdot\vec{b}'(t_0)+\vec{a}'(t_0)\cdot\vec{b}(t_0)$$

$$lemma\ 2:\left(\vec{a}(t_0)\times\vec{b}(t_0)\right)'=\vec{a}(t_0)\times\vec{b}'(t_0)+\vec{a}'(t_0)\times\vec{b}(t_0)$$

$$proof:\left(\vec{a}(t_0)\times\vec{b}(t_0)\right)'=\lim_{t\to t_0}\frac{\vec{a}(t)\times\vec{b}(t)-\vec{a}(t_0)\times\vec{b}(t_0)}{t-t_0}$$

$$=\lim_{t\to t_0}\frac{\left[\vec{a}(t)\times\vec{b}(t)-\vec{a}(t)\times\vec{b}(t_0)\right]+\left[\vec{a}(t)\times\vec{b}(t_0)-\vec{a}(t_0)\times\vec{b}(t_0)\right]}{t-t_0}$$

$$=\lim_{t\to t_0}\vec{a}(t)\times\frac{\vec{b}(t)-\vec{b}(t_0)}{t-t_0}+\lim_{t\to t_0}\frac{\vec{a}(t)\times\vec{b}(t_0)-\vec{a}(t_0)\times\vec{b}(t_0)}{t-t_0}$$

$$=\lim_{t\to t_0}\vec{a}(t)\times\lim_{t\to t_0}\frac{\vec{b}(t)-\vec{b}(t_0)}{t-t_0}+\lim_{t\to t_0}\frac{\vec{a}(t)-\vec{a}(t_0)}{t-t_0}\times\vec{b}(t_0)$$

$$=\vec{a}(t_0)\times\vec{b}'(t_0)+\vec{a}'(t_0)\times\vec{b}(t_0)$$

$$then:\frac{d\left(\vec{r}_1(t),\vec{r}_2(t),\vec{r}_3(t)\right)}{dt}=\frac{d\left(\vec{r}_1(t)\cdot\left(\vec{r}_2(t)\times\vec{r}_3(t)\right)\right)}{dt}$$

$$=\frac{d\left(\vec{r}_1(t)\right)}{dt}\cdot\left(\vec{r}_2(t)\times\vec{r}_3(t)\right)+\vec{r}_1(t)\cdot\frac{d\left(\vec{r}_2(t)\times\vec{r}_3(t)\right)}{dt}$$

$$=\frac{d\left(\vec{r}_1(t)\right)}{dt}\cdot\left(\vec{r}_2(t)\times\vec{r}_3(t)\right)+\vec{r}_1(t)\cdot\left[\frac{d\left(\vec{r}_2(t)\right)}{dt}\times\vec{r}_3(t)+\vec{r}_2(t)\times\frac{d\left(\vec{r}_3(t)\right)}{dt}\right]$$

$$=\frac{d\left(\vec{r}_1(t)\right)}{dt}\cdot\left(\vec{r}_2(t)\times\vec{r}_3(t)\right)+\vec{r}_1(t)\cdot\frac{d\left(\vec{r}_2(t)\right)}{dt}\times\vec{r}_3(t)+\vec{r}_1(t)\cdot\vec{r}_2(t)\times\frac{d\left(\vec{r}_3(t)\right)}{dt}$$

$$=\left(\vec{r}_1'(t),\vec{r}_2(t),\vec{r}_3(t)\right)+\left(\vec{r}_1(t),\vec{r}_2'(t),\vec{r}_3(t)\right)+\left(\vec{r}_1(t),\vec{r}_2(t),\vec{r}_3'(t)\right)$$

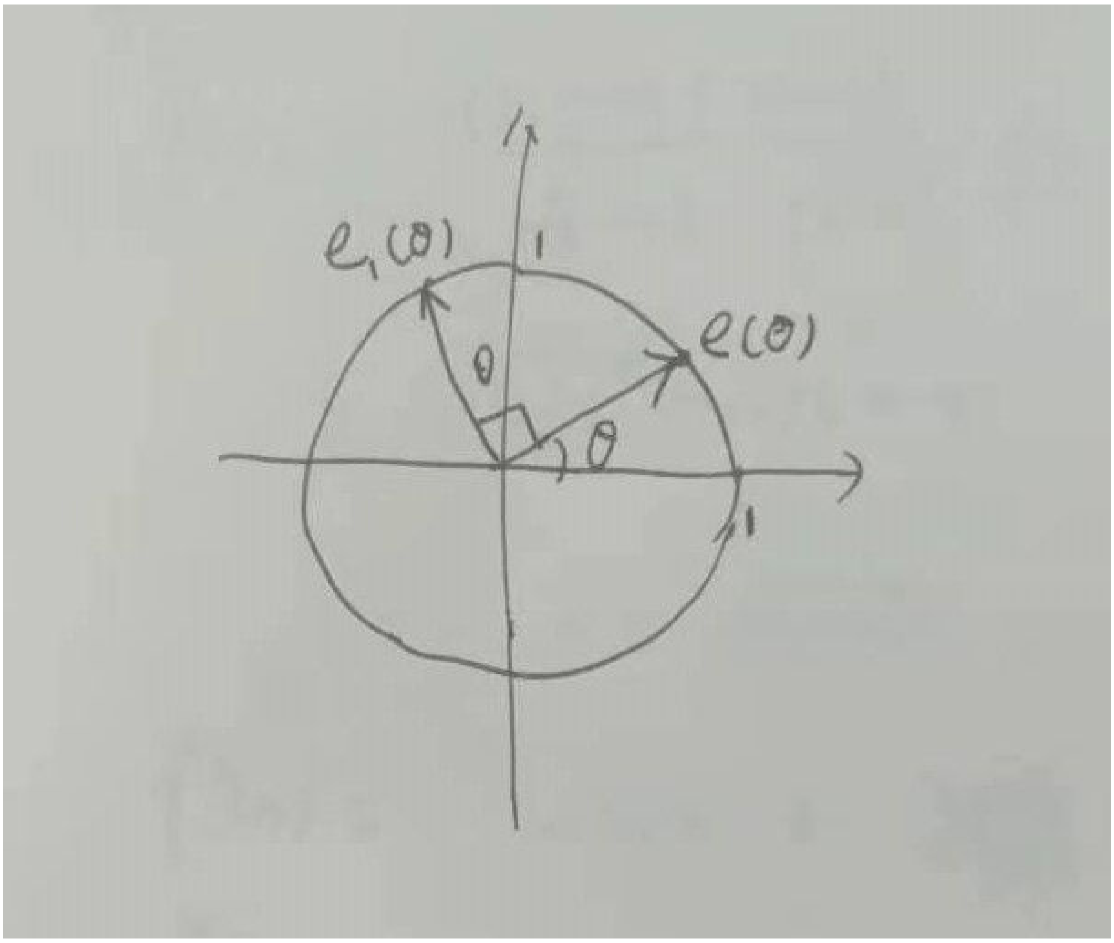
$$2.(2)\frac{d\left(\vec{r}_1(t),\vec{r}_2(t),\vec{r}_3(t)\right)}{dt}=\frac{d\left(\vec{r}_1(t)\cdot\left(\vec{r}_2(t)\times\vec{r}_3(t)\right)\right)}{dt}$$

choose $\vec{r}_1(t)=\vec{r}(t),\vec{r}_2(t)=\vec{r}\,'(t),\vec{r}_3(t)=\vec{r}\,''(t)$

$$by\ 2.(1):\frac{d\left(\vec{r}(t),\vec{r}\,'(t),\vec{r}\,''(t)\right)}{dt}$$

$$=\left(\vec{r}\,'(t),\vec{r}\,'(t),\vec{r}\,''(t)\right)+\left(\vec{r}(t),\vec{r}\,''(t),\vec{r}\,''(t)\right)+\left(\vec{r}(t),\vec{r}\,'(t),\vec{r}\,'''(t)\right)$$

$$=0+0+\left(\vec{r}(t),\vec{r}\,'(t),\vec{r}\,'''(t)\right)=\left(\vec{r}(t),\vec{r}\,'(t),\vec{r}\,'''(t)\right)$$



3.(2) *proof*:

$$\vec{e}\,'(\theta)=\left(\cos\theta\vec{i}+\sin\theta\vec{j}\right)'=-\sin\theta\vec{i}+\cos\theta\vec{j}=\vec{e}_1(\theta)$$

$$\vec{e}_1\,'(\theta)=\left(-\sin\theta\vec{i}+\cos\theta\vec{j}\right)'=-\left(\cos\theta\vec{i}+\sin\theta\vec{j}\right)=-\vec{e}(\theta)$$

3.(3) *proof*:

choose a point on C *denoted by* (r,θ)

$$\vec{r}\cdot\vec{i}=r\cos\theta,\vec{r}\cdot\vec{j}=r\sin\theta$$

$$\Rightarrow\vec{r}=r(\theta)\vec{e}(\theta)$$

$$\vec{r}\,'=r'(\theta)\vec{e}(\theta)+r(\theta)\vec{e}\,'(\theta)=\left(r'(\theta)\cos\theta-r(\theta)\sin\theta\right)\vec{i}+\left(r'(\theta)\sin\theta+r(\theta)\cos\theta\right)\vec{j}$$

4.(1) *proof* :

$$\vec{a}(t)\cdot\vec{b}(t)=0\Rightarrow\frac{d\left(\vec{a}(t)\cdot\vec{b}(t)\right)}{dt}=0$$

$$\Rightarrow\vec{a}'(t)\cdot\vec{b}(t)+\vec{a}(t)\cdot\vec{b}'(t)=0\Rightarrow\vec{a}'(t)\cdot\vec{b}(t)=-\vec{a}(t)\cdot\vec{b}'(t)\triangleq k(t)$$

$$\Rightarrow\begin{cases}\vec{a}'(t)\cdot\vec{b}(t)=k(t)\\ \vec{a}(t)\cdot\vec{b}'(t)=-k(t)\end{cases}$$

$$\Rightarrow\begin{cases}\vec{a}'(t)\cdot\vec{b}(t)\cdot\vec{b}(t)=k(t)\cdot\vec{b}(t)\\ \vec{a}(t)\cdot\vec{a}(t)\cdot\vec{b}'(t)=-\vec{a}(t)\cdot k(t)\end{cases}$$

$$\Rightarrow\begin{cases}\vec{a}'(t)\cdot\left(\vec{b}(t)\cdot\vec{b}(t)\right)=k(t)\cdot\vec{b}(t)\\ \left(\vec{a}(t)\cdot\vec{a}(t)\right)\cdot\vec{b}'(t)=-\vec{a}(t)\cdot k(t)\end{cases}$$

$$\Rightarrow\begin{cases}\vec{a}'(t)=k(t)\cdot\vec{b}(t)\\ \vec{b}'(t)=-\vec{a}(t)\cdot k(t)\end{cases}$$

$$k(t)=\vec{a}'(t)\cdot\vec{b}(t)$$

4.(2) *proof* :

$$\vec{a}(t)\cdot\vec{b}(t)=\vec{b}(t)\cdot\vec{c}(t)=\vec{c}(t)\cdot\vec{a}(t)=0$$

$$\Rightarrow\begin{cases}\vec{a}'(t)\cdot\vec{b}(t)+\vec{a}(t)\cdot\vec{b}'(t)=0\Rightarrow\vec{a}'(t)\cdot\vec{b}(t)=-\vec{a}(t)\cdot\vec{b}'(t)\triangleq 2k_1(t)\\ \vec{c}'(t)\cdot\vec{a}(t)+\vec{c}(t)\cdot\vec{a}'(t)=0\Rightarrow-\vec{c}'(t)\cdot\vec{a}(t)=\vec{c}(t)\cdot\vec{a}'(t)\triangleq 2k_2(t)\\ \vec{b}'(t)\cdot\vec{c}(t)+\vec{b}(t)\cdot\vec{c}'(t)=0\Rightarrow\vec{b}'(t)\cdot\vec{c}(t)=-\vec{b}(t)\cdot\vec{c}'(t)\triangleq 2k_3(t)\end{cases}$$

$$\Rightarrow\begin{cases}\vec{a}'(t)=\vec{a}'(t)\cdot\vec{b}(t)\cdot\vec{b}(t)=2k_1(t)\cdot\vec{b}(t)\\ \vec{a}'(t)=\vec{c}(t)\cdot\vec{c}(t)\cdot\vec{a}'(t)=2\vec{c}(t)\cdot k_2(t)=2k_2(t)\cdot\vec{c}(t)\end{cases}$$

$$\Rightarrow 2\vec{a}'(t)=2k_1(t)\cdot\vec{b}(t)+2k_2(t)\cdot\vec{c}(t)$$

$$\Rightarrow\vec{a}'(t)=k_1(t)\cdot\vec{b}(t)+k_2(t)\cdot\vec{c}(t)$$

$$similarly,\vec{b}'(t)=-k_1(t)\cdot\vec{a}(t)+k_3(t)\cdot\vec{c}(t),\vec{c}'(t)=-k_2(t)\cdot\vec{a}(t)-k_3(t)\cdot\vec{b}(t)$$

$$k_1(t)=\frac{\vec{a}'(t)\cdot\vec{b}(t)}{2},k_2(t)=\frac{\vec{c}(t)\cdot\vec{a}'(t)}{2},k_3(t)=\frac{\vec{b}'(t)\cdot\vec{c}(t)}{2}$$

5.*proof* :

$$\vec{r}=a\cos\theta\vec{i}+a\sin\theta\vec{j}+b\theta\vec{k}$$

$$\vec{r}'=\left(a\cos\theta\vec{i}+a\sin\theta\vec{j}+b\theta\vec{k}\right)'=-a\sin\theta\vec{i}+a\cos\theta\vec{j}+b\vec{k}$$

$$\cos\left\langle\vec{r}',\vec{k}\right\rangle=\frac{\vec{r}'\cdot\vec{k}}{\left|\vec{r}'\right|\cdot\left|\vec{k}\right|}=\frac{b}{\sqrt{(a\cos\theta)^2+(a\sin\theta)^2+b^2}}=\frac{b}{\sqrt{a^2+b^2}}$$

补充题：

记$z(t)=x(t)+iy(t)$其中$x(t),y(t)$为实值函数.

由于复值函数$z(t)$在t_0处可导, 这说明$z(t)=x(t)+iy(t)$在t_0处连续，

这说明$x(t),y(t)$在t_0处连续.

同时：

$$z'(t_0)=\lim_{t\to t_0}\frac{x(t)+iy(t)-x(t_0)-iy(t_0)}{t-t_0}$$

$$=\lim_{t\to t_0}\frac{x(t)-x(t_0)}{t-t_0}+i\lim_{t\to t_0}\frac{y(t)-y(t_0)}{t-t_0}\text{存在}$$

这说明$x'(t_0)=\lim_{t\to t_0}\frac{x(t)-x(t_0)}{t-t_0},y'(t_0)=\lim_{t\to t_0}\frac{y(t)-y(t_0)}{t-t_0}$存在

$$|z|(t)=|z(t)|=\sqrt{x^2(t)+y^2(t)}$$

$$|z|'(t_0)=\lim_{t\to t_0}\frac{\sqrt{x^2(t)+y^2(t)}-\sqrt{x^2(t_0)+y^2(t_0)}}{t-t_0}$$

$$=\lim_{t\to t_0}\frac{x^2(t)-x^2(t_0)+y^2(t)-y^2(t_0)}{t-t_0}\frac{1}{\sqrt{x^2(t)+y^2(t)}+\sqrt{x^2(t_0)+y^2(t_0)}}$$

$$=\frac{\lim\limits_{t\to t_0}\frac{x(t)-x(t_0)}{t-t_0}\lim\limits_{t\to t_0}\left(x(t)+x(t_0)\right)+\lim\limits_{t\to t_0}\frac{y(t)-y(t_0)}{t-t_0}\lim\limits_{t\to t_0}\left(y(t)+y(t_0)\right)}{\lim\limits_{t\to t_0}\left[\sqrt{x^2(t)+y^2(t)}+\sqrt{x^2(t_0)+y^2(t_0)}\right]}$$

$$=\left[2x'(t_0)x(t_0)+2y'(t_0)y(t_0)\right]\frac{1}{2\sqrt{x^2(t_0)+y^2(t_0)}}$$

$$=\frac{x'(t_0)x(t_0)+y'(t_0)y(t_0)}{\sqrt{x^2(t_0)+y^2(t_0)}}$$

而$\frac{z'(t_0)\overline{z(t_0)}+z(t_0)\overline{z'(t_0)}}{2|z(t_0)|}$

$$=\frac{\left(x'(t_0)+iy'(t_0)\right)\left(x(t_0)-iy(t_0)\right)+\left(x(t_0)+iy(t_0)\right)\left(x'(t_0)-iy'(t_0)\right)}{2\sqrt{x^2(t_0)+y^2(t_0)}}$$

$$=\frac{x'(t_0)x(t_0)+y'(t_0)y(t_0)}{\sqrt{x^2(t_0)+y^2(t_0)}}=|z|'(t_0)$$

•当$x(t_0)=y(t_0)=0$时，不妨设$t_0=0$,否则用$z(t-t_0)$代替$z(t)$.

则$x(0)=y(0)=0$

$$|z|'(0+)=\lim_{t\to 0^+}\frac{\sqrt{x^2(t)+y^2(t)}}{t}$$

$$|z|'(0-)=\lim_{t\to 0^-}\frac{\sqrt{x^2(t)+y^2(t)}}{t}$$

如果我们选定$x(t)=y(t)=t^{\frac{1}{3}}$

则$|z|'(0+)=\lim\limits_{t\to 0^+}\frac{\sqrt{x^2(t)+y^2(t)}}{t}=\lim\limits_{t\to 0^+}\frac{\sqrt{2t^{\frac{2}{3}}}}{t}=\lim\limits_{t\to 0^+}\frac{\sqrt{2}t^{\frac{1}{3}}}{t}=\lim\limits_{t\to 0^+}\frac{\sqrt{2}}{t^{\frac{2}{3}}}=+\infty$

$$|z|'(0-)=\lim_{t\to 0^-}\frac{\sqrt{x^2(t)+y^2(t)}}{t}=\lim_{t\to 0^-}\frac{\sqrt{2t^{\frac{2}{3}}}}{t}=\lim_{t\to 0^-}\frac{\sqrt{2}|t|^{\frac{1}{3}}}{t}=-\lim_{t\to 0^-}\frac{\sqrt{2}}{t^{\frac{2}{3}}}=-\infty\neq|z|'(0+)$$

因此,$|z|'(t_0)$不存在.

$1.proof:$

当$\lim\limits_{x\to a^+}f\left(x\right)=\lim\limits_{x\to b^-}f\left(x\right)\in\mathbb{R}$时，平凡

当$\lim\limits_{x\to a^+}f\left(x\right)=\lim\limits_{x\to b^-}f\left(x\right)=\infty$时，

不妨设$\lim\limits_{x\to a^+}f\left(x\right)=\lim\limits_{x\to b^-}f\left(x\right)=+\infty$

$\Rightarrow$对于一个给定的$x_0\in\left(a,b\right)$,

$\exists\delta>0,\exists a$在$\left(a,b\right)$的一个开邻域$U_a$,$b$在$\left(a,b\right)$的一个开邻域$U_b$, s.t.

$\forall x\in U_a\cup U_b$,有$f\left(x\right)>f\left(x_0\right)$

•如果$\forall x\in\left(a,b\right)$,有$f\left(x\right)\geq f\left(x_0\right)$

所以x_0是f的极值点，又因为f在$\left(a,b\right)$上处处可微，因此f在x_0处可导

由费马定理$f'\left(x_0\right)=0$

•如果$\exists x\in\left(a,b\right)$,有$f\left(x\right)<f\left(x_0\right)$,那么$U_a\cup U_b\subseteq\left(a,b\right)$, 记$V=\left(a,b\right)\backslash\left(U_a\cup U_b\right)\neq\varnothing$

取一点$x_1\in V$,则有$f\left(x_1\right)<f\left(x_0\right)$，不妨考虑$x_1<x_0$的情况，

再取一点$x_2\in U_a\cap\left(a,x_1\right)$,则有$f\left(x_2\right)>f\left(x_0\right)$,

因为$\left(x_1,x_2\right)\subset\left(a,b\right)$, f在$\left(a,b\right)$上连续，所以f在$\left(x_1,x_2\right)$上连续.

由连续函数的介值定理:对于$f\left(x_0\right)\in\left(f\left(x_1\right),f\left(x_2\right)\right)$,$\exists x_3\in\left(x_2,x_1\right)$, $s.t.$

$f\left(x_3\right)=f\left(x_0\right)$，而$x_3<x_1<x_0$,又因为$\left[x_3,x_0\right]\subset\left(a,b\right)$, f在$\left[x_3,x_0\right]$上连续可微

由普通的罗尔定理, $\exists\xi\in\left(x_3,x_0\right)\subset\left(a,b\right)$， $s.t.$

$f'\left(\xi\right)=0$

得证！

$3.\left(1\right)proof:$

$f\left(x\right)\triangleq\frac{a}{4}x^4+\frac{b}{3}x^3+\frac{c}{2}x^2, f'\left(x\right)=ax^3+bx^2+cx, f\left(1\right)=\frac{a}{4}+\frac{b}{3}+\frac{c}{2}, f\left(0\right)=0$

只需证:$\exists x_0\in\left(0,1\right), s.t. f'\left(x_0\right)=f\left(1\right)$

$i.e. f'\left(x_0\right)=\dfrac{f\left(1\right)-f\left(0\right)}{1-0}$

这一点由拉格朗日中值定理保证，得证！

$3.\left(2\right)proof:$

$f\left(x\right)\triangleq x^3+ax^2+bx+c, f'\left(x\right)=3x^2+2ax+b=3\left(x+\frac{a}{3}\right)^2+b-\frac{a^2}{3}>0$

$\forall x>y,\exists\xi\in\left(y,x\right), f\left(x\right)-f\left(y\right)=f'\left(\xi\right)\left(x-y\right)>0,$

f is monotonically increasing!

$\lim\limits_{x\to-\infty}f\left(x\right)=-\infty, \lim\limits_{x\to+\infty}f\left(x\right)=+\infty$

by intermediate value theorem:

$\exists! x_0\in\mathbb{R}, s.t. f\left(x_0\right)=0.$

$5.proof:$

我们反证:

假设$\exists f(x)$的两个相邻的零点$x_1,x_2,\forall x\in(x_1,x_2)\subset I,g(x)\neq 0.$

由连续函数介值定理，我们不妨设$\forall x\in(x_1,x_2)\subset I,g(x)>0.$

记$h(x)=\dfrac{f(x)}{g(x)},x\in I,h'(x)=\dfrac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}.$

$h(x_1)=\dfrac{f(x_1)}{g(x_1)}=0,h(x_2)=\dfrac{f(x_2)}{g(x_2)}=0$

$\forall x\in I,h'(x)\neq 0.$

由拉格朗日中值定理:

$\exists \xi\in(x_1,x_2)\subset I,s.t.h'(\xi)=\dfrac{h(x_1)-h(x_2)}{x_1-x_2}=0,$矛盾!

故在$f(x)$的任意两个零点之间都夹有$g(x)$的一个零点.

$6.(1)\,proof:$

记$h(x)=e^{ax}f(x),x\in I,h'(x)=e^{ax}\left[af(x)+f'(x)\right]$

$h(x_1)=e^{ax_1}f(x_1)=0,h(x_2)=e^{ax_2}f(x_2)=0.$

由拉格朗日中值定理:

$\exists \xi\in(x_1,x_2),s.t.h'(\xi)=\dfrac{h(x_1)-h(x_2)}{x_1-x_2}=0,i.e.\,af(\xi)+f'(\xi)=0$

故在$f(x)$的任意两个零点之间都夹有$f'(x)+af(x)$的一个零点.

$6.(2)\,proof:$

记$h(x)=x^a f(x),x\in(x_1,x_2)\subset(0,+\infty),h'(x)=x^a f(x)+ax^{a-1}f'(x).$

$h(x_1)=x_1^a f(x_1)=0,h(x_2)=x_2^a f(x_2)=0.$

由拉格朗日中值定理:

$\exists \xi\in(x_1,x_2)\subset(0,+\infty),s.t.h'(\xi)=\dfrac{h(x_1)-h(x_2)}{x_1-x_2}=0,i.e.\,\xi^a f(\xi)+a\xi^{a-1}f'(\xi)=0$

$\Rightarrow \xi f'(\xi)+af(\xi)=0$

故在$f(x)$的任意两个零点之间都夹有$xf'(x)+af(x)$的一个零点.

$9.(1)\ proof:$

取$f(x)=\arctan x(x>0), f'(x)=\dfrac{1}{1+x^2}, f''(x)=-\dfrac{2x}{(1+x^2)^2}<0$, 故$f'(x)$严格单调递减.

$\exists\xi\in(a,b)\subset(0,+\infty), s.t. f'(\xi)=\dfrac{f(a)-f(b)}{a-b}$

$f'(b)<f'(\xi)<f'(a)$

$\Rightarrow f'(b)<\dfrac{f(a)-f(b)}{a-b}<f'(a)$

$\Rightarrow \dfrac{a-b}{1+a^2}<\arctan a-\arctan b<\dfrac{a-b}{1+b^2}$

$9.(2)\ proof:$

取$f(x)=\ln x(x>0), f'(x)=\dfrac{1}{x}, f''(x)=-\dfrac{1}{x^2}<0$, 故$f'(x)$严格单调递减.

$\exists\xi\in(a,b)\subset(0,+\infty), s.t. f'(\xi)=\dfrac{f(a)-f(b)}{a-b}$

$f'(b)<f'(\xi)<f'(a)$

$\Rightarrow f'(b)<\dfrac{f(a)-f(b)}{a-b}<f'(a)$

$\Rightarrow \dfrac{a-b}{a}<\ln\dfrac{a}{b}<\dfrac{a-b}{b}.$

$9.(1)\ proof:$

取$f(x)=x^p(x>0), f'(x)=px^{p-1}, f''(x)=p(p-1)x^{p-2}>0$, 故$f'(x)$严格单调递增.

$a\neq b$时,

$\exists\xi\in(a,b)\subset(0,+\infty), s.t. f'(\xi)=\dfrac{f(a)-f(b)}{a-b}$

$f'(b)>f'(\xi)>f'(a)$

$\Rightarrow f'(b)>\dfrac{f(a)-f(b)}{a-b}>f'(a)$

$\Rightarrow pb^{p-1}(a-b)<a^p-b^p<pa^{p-1}(a-b).$

$a=b$时，$pb^{p-1}(a-b)=a^p-b^p=pa^{p-1}(a-b)=0$

综上，$\forall a,b>0, p>1$, 我们有$pb^{p-1}(a-b)\le a^p-b^p\le pa^{p-1}(a-b)$