10/30 homework

$$1.(1)\lim_{\Delta x\to 0}\frac{\left[(x+\Delta x)^3+2(x+\Delta x)\right]-(x^3+2x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3+2\Delta x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}3x^2+3x\Delta x+(\Delta x)^2+2=3x^2+2$$

$$1.(2)\lim_{\Delta x\to 0}\frac{\frac{x+\Delta x}{x+\Delta x-1}-\frac{x}{x-1}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{1}{x+\Delta x-1}-\frac{1}{x-1}}{\Delta x}=\lim_{\Delta x\to 0}\frac{-\Delta x}{\Delta x(x+\Delta x-1)(x-1)}$$

$$=-\lim_{\Delta x\to 0}\frac{1}{(x+\Delta x-1)(x-1)}=-\frac{1}{(x-1)^2}$$

$$2.(2)\,lemma:\lim_{x\to 0}\frac{\arctan x}{x}=1,$$

$$proof:\lim_{x\to 0}\frac{\arctan x}{x}=\lim_{x\to 0}\frac{x}{\tan x}=\lim_{x\to 0}\frac{x\cos x}{\sin x}=\lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}\cos x=1$$

$$\lim_{\Delta x\to 0}\frac{\arctan(x+\Delta x)-\arctan x}{\Delta x}=\lim_{\Delta x\to 0}\frac{\arctan\left(\frac{x+\Delta x-x}{1+(x+\Delta x)x}\right)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\arctan\left(\frac{\Delta x}{1+(x+\Delta x)x}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\arctan\left(\frac{\Delta x}{1+(x+\Delta x)x}\right)}{\frac{\Delta x}{1+(x+\Delta x)x}}\cdot\frac{\frac{\Delta x}{1+(x+\Delta x)x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{1+(x+\Delta x)x}=\frac{1}{1+x^2}$$

$$2.(3)\lim_{\Delta x\to 0}\frac{\arcsin(x+\Delta x)-\arcsin x}{\Delta x}=\lim_{\Delta x\to 0}\frac{(x+\Delta x)\sqrt{1-x^2}-x\sqrt{1-(x+\Delta x)^2}}{\Delta x}$$

$$=\sqrt{1-x^2}\lim_{\Delta x\to 0}\frac{(x+\Delta x)-x\sqrt{\frac{1-(x+\Delta x)^2}{1-x^2}}}{\Delta x}=\sqrt{1-x^2}\lim_{\Delta x\to 0}\frac{(x+\Delta x)-x\sqrt{1-\frac{2x\Delta x+(\Delta x)^2}{1-x^2}}}{\Delta x}$$

$$=\sqrt{1-x^2}\left[1+\lim_{\Delta x\to 0}\frac{x-x\sqrt{1-\frac{2x\Delta x+(\Delta x)^2}{1-x^2}}}{\Delta x}\right]=\sqrt{1-x^2}\left[1+x\lim_{\Delta x\to 0}\frac{1-\sqrt{1-\frac{2x\Delta x+(\Delta x)^2}{1-x^2}}}{\Delta x}\right]$$

$$=\sqrt{1-x^2}\left[1+x\lim_{\Delta x\to 0}\frac{1-\left(1-\frac{2x\Delta x+(\Delta x)^2}{1-x^2}\right)}{\Delta x\left(1+\sqrt{1-\frac{2x\Delta x+(\Delta x)^2}{1-x^2}}\right)}\right]=\sqrt{1-x^2}\left[1+x\lim_{\Delta x\to 0}\frac{\frac{2x\Delta x+(\Delta x)^2}{1-x^2}}{\Delta x\left(1+\sqrt{1-\frac{2x\Delta x+(\Delta x)^2}{1-x^2}}\right)}\right]$$

$$=\sqrt{1-x^2}\left[1+x\frac{\lim_{\Delta x\to 0}\frac{2x+\Delta x}{1-x^2}}{\lim_{\Delta x\to 0}\left(1+\sqrt{1-\frac{2x\Delta x+(\Delta x)^2}{1-x^2}}\right)}\right]=\sqrt{1-x^2}\left(1+\frac{x^2}{1-x^2}\right)=\frac{1}{\sqrt{1-x^2}}$$

$$3.(2)\,f(x)=e^x\sin(x-1)$$

$$f'(1)=\lim_{\Delta x\to 0}\frac{f(1+\Delta x)-f(1)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(1+\Delta x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{e^{1+\Delta x}\sin(\Delta x)}{\Delta x}=\lim_{\Delta x\to 0}e^{1+\Delta x}\lim_{\Delta x\to 0}\frac{\sin(\Delta x)}{\Delta x}=e$$

$$3.(3)\,f(x)=x^2+\arccos\sqrt[3]{1-x^2}\ln x$$

$$f'(1)=\lim_{\Delta x\to 0}\frac{f(1+\Delta x)-f(1)}{\Delta x}=\lim_{\Delta x\to 0}\frac{(1+\Delta x)^2+\arccos\sqrt[3]{1-(1+\Delta x)^2}\ln(1+\Delta x)-1}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{2\Delta x+(\Delta x)^2+\arccos\sqrt[3]{-2\Delta x-(\Delta x)^2}\ln(1+\Delta x)}{\Delta x}=2+\lim_{\Delta x\to 0}\arccos\sqrt[3]{-2\Delta x-(\Delta x)^2}\cdot\lim_{\Delta x\to 0}\frac{\ln(1+\Delta x)}{\Delta x}$$

$$=2+\arccos 0=2+\frac{\pi}{2}$$

$$4.(2)\, f(x)=\begin{cases}2^{-\frac{1}{x}}, x>0\\ 0, x\le 0\end{cases}$$

$$\lim_{x\to 0^+} f'(x)=\lim_{\Delta x\to 0^+}\frac{f(\Delta x)-f(0)}{\Delta x}=\lim_{\Delta x\to 0^+}\frac{2^{-\frac{1}{\Delta x}}}{\Delta x}\overset{y=-\frac{1}{\Delta x}\to-\infty}{=}\lim_{y\to-\infty}(-y)2^y=0$$

$$\lim_{x\to 0^-} f'(x)=\lim_{\Delta x\to 0^-}\frac{f(\Delta x)-f(0)}{\Delta x}=0=\lim_{x\to 0^+} f'(x)$$

$$\Rightarrow \lim_{x\to 0} f'(x)\, exists$$

$$f'(x)=\lim_{x\to 0} f'(x)=\lim_{x\to 0^-} f'(x)=\lim_{x\to 0^+} f'(x)=0$$

$$4.(3)\, f(x)=\begin{cases}x^2, x\in\mathbb{Q}\\ 0, x\notin\mathbb{Q}\end{cases}$$

$$\lim_{x\to 0\wedge x\in\mathbb{Q}} f'(x)=\lim_{\Delta x\to 0\wedge x\in\mathbb{Q}}\frac{f(\Delta x)-f(0)}{\Delta x}=\lim_{\Delta x\to 0\wedge x\in\mathbb{Q}}\Delta x=0$$

$$\lim_{x\to 0\wedge x\notin\mathbb{Q}} f'(x)=\lim_{\Delta x\to 0\wedge x\notin\mathbb{Q}}\frac{f(\Delta x)-f(0)}{\Delta x}=\lim_{\Delta x\to 0\wedge x\in\mathbb{Q}}0=0=\lim_{x\to 0\wedge x\in\mathbb{Q}} f'(x)$$

$$\Rightarrow \lim_{x\to 0} f'(x)\, exists.$$

$$\lim_{x\to 0} f'(x)=\lim_{x\to 0\wedge x\in\mathbb{Q}} f'(x)=\lim_{x\to 0\wedge x\notin\mathbb{Q}} f'(x)=0$$

$$6.(1)\, y=2x^3+3x^2+6x$$

$$y'=\lim_{\Delta x\to 0}\frac{\left(2(x+\Delta x)^3+3(x+\Delta x)^2+6(x+\Delta x)\right)-\left(2x^3+3x^2+6x\right)}{\Delta x}$$

$$=2\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x}+3\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}+6\lim_{\Delta x\to 0}\frac{(x+\Delta x)-x}{\Delta x}$$

$$=2\lim_{\Delta x\to 0}\frac{3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3}{\Delta x}+3\lim_{\Delta x\to 0}\frac{2x\Delta x+(\Delta x)^2}{\Delta x}+6\lim_{\Delta x\to 0}\frac{(x+\Delta x)-x}{\Delta x}$$

$$=6x^2+6x+6$$

$$6.(4)\, y=\frac{1}{\sqrt{x}}+\frac{1}{\sqrt[3]{x}}+\frac{1}{\sqrt[6]{x}}$$

$$y'=\lim_{\Delta x\to 0}\frac{\left(\frac{1}{\sqrt{x+\Delta x}}+\frac{1}{\sqrt[3]{x+\Delta x}}+\frac{1}{\sqrt[6]{x+\Delta x}}\right)-\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt[3]{x}}+\frac{1}{\sqrt[6]{x}}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\frac{1}{\sqrt{x+\Delta x}}-\frac{1}{\sqrt{x}}}{\Delta x}+\lim_{\Delta x\to 0}\frac{\frac{1}{\sqrt[3]{x+\Delta x}}-\frac{1}{\sqrt[3]{x}}}{\Delta x}+\lim_{\Delta x\to 0}\frac{\frac{1}{\sqrt[6]{x+\Delta x}}-\frac{1}{\sqrt[6]{x}}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sqrt{x}-\sqrt{x+\Delta x}}{\Delta x\sqrt{x+\Delta x}\sqrt{x}}+\lim_{\Delta x\to 0}\frac{\sqrt[3]{x}-\sqrt[3]{x+\Delta x}}{\Delta x\sqrt[3]{x+\Delta x}\sqrt[3]{x}}+\lim_{\Delta x\to 0}\frac{\sqrt[6]{x}-\sqrt[6]{x+\Delta x}}{\Delta x\sqrt[6]{x+\Delta x}\sqrt[6]{x}}$$

$$=\lim_{\Delta x\to 0}\frac{-\Delta x}{\Delta x\sqrt{x+\Delta x}\sqrt{x}\left(\sqrt{x}+\sqrt{x+\Delta x}\right)}+\lim_{\Delta x\to 0}\frac{-\Delta x}{\Delta x\sqrt[3]{x+\Delta x}\sqrt[3]{x}\left(\sqrt[3]{x^2}+\sqrt[3]{x(x+\Delta x)}+\sqrt[3]{(x+\Delta x)^2}\right)}$$

$$+\lim_{\Delta x\to 0}\frac{-\Delta x}{\Delta x\sqrt[6]{x+\Delta x}\sqrt[6]{x}\left(\sqrt[6]{x^5}+\sqrt[6]{x^4(x+\Delta x)}+\sqrt[6]{x^3(x+\Delta x)^2}+\sqrt[6]{x^2(x+\Delta x)^3}+\sqrt[6]{x(x+\Delta x)^5}+\sqrt[6]{(x+\Delta x)^5}\right)}$$

$$=-\lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}\sqrt{x}\left(\sqrt{x}+\sqrt{x+\Delta x}\right)}-\lim_{\Delta x\to 0}\frac{1}{\sqrt[3]{x+\Delta x}\sqrt[3]{x}\left(\sqrt[3]{x^2}+\sqrt[3]{x(x+\Delta x)}+\sqrt[3]{(x+\Delta x)^2}\right)}$$

$$-\lim_{\Delta x\to 0}\frac{1}{\sqrt[6]{x+\Delta x}\sqrt[6]{x}\left(\sqrt[6]{x^5}+\sqrt[6]{x^4(x+\Delta x)}+\sqrt[6]{x^3(x+\Delta x)^2}+\sqrt[6]{x^2(x+\Delta x)^3}+\sqrt[6]{x(x+\Delta x)^5}+\sqrt[6]{(x+\Delta x)^5}\right)}$$

$$=-\frac{1}{\lim_{\Delta x\to 0}\sqrt{x+\Delta x}\sqrt{x}\left(\sqrt{x}+\sqrt{x+\Delta x}\right)}-\frac{1}{\lim_{\Delta x\to 0}\sqrt[3]{x+\Delta x}\sqrt[3]{x}\left(\sqrt[3]{x^2}+\sqrt[3]{x(x+\Delta x)}+\sqrt[3]{(x+\Delta x)^2}\right)}$$

$$-\frac{1}{\lim_{\Delta x\to 0}\sqrt[6]{x+\Delta x}\sqrt[6]{x}\left(\sqrt[6]{x^5}+\sqrt[6]{x^4(x+\Delta x)}+\sqrt[6]{x^3(x+\Delta x)^2}+\sqrt[6]{x^2(x+\Delta x)^3}+\sqrt[6]{x(x+\Delta x)^5}+\sqrt[6]{(x+\Delta x)^5}\right)}$$

$$=-\frac{1}{2\sqrt{x^3}}-\frac{1}{3\sqrt[3]{x^4}}-\frac{1}{6\sqrt[6]{x^7}}=-\frac{1}{2}x^{-\frac{3}{2}}-\frac{1}{3}x^{-\frac{4}{3}}-\frac{1}{6}x^{-\frac{7}{6}}$$

$6.(5)\ y=x^2\sin x+x\cos x$

$$y'=\lim_{\Delta x\to 0}\frac{\left((x+\Delta x)^2\sin(x+\Delta x)+(x+\Delta x)\cos(x+\Delta x)\right)-\left(x^2\sin x+x\cos x\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2\sin(x+\Delta x)-x^2\sin x}{\Delta x}+\lim_{\Delta x\to 0}\frac{(x+\Delta x)\cos(x+\Delta x)-x\cos x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{x^2\left(\sin(x+\Delta x)-\sin x\right)+2x\Delta x\sin(x+\Delta x)+(\Delta x)^2\sin(x+\Delta x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{x\left(\cos(x+\Delta x)-\cos x\right)-\Delta x\cos(x+\Delta x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{2x^2\cos\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}+2x\Delta x\sin(x+\Delta x)+(\Delta x)^2\sin(x+\Delta x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{2x\sin\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}-\Delta x\cos(x+\Delta x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\left[x^2\cos\left(x+\frac{\Delta x}{2}\right)+2x\sin(x+\Delta x)+\Delta x\sin(x+\Delta x)\right]+\lim_{\Delta x\to 0}\left[x\sin\left(x+\frac{\Delta x}{2}\right)-\cos(x+\Delta x)\right]$$

$$=x^2\cos x+2x\sin x+x\sin x-\cos x=x^2\cos x+3x\sin x-\cos x$$

$6.(6)\ y=\left(x^3+x^2-x\right)\ln x$

$$y'=\lim_{\Delta x\to 0}\frac{\left((x+\Delta x)^3+(x+\Delta x)^2-(x+\Delta x)\right)-\left(x^3+x^2-x\right)}{\Delta x}\ln x+\left(x^3+x^2-x\right)\lim_{\Delta x\to 0}\frac{\ln(x+\Delta x)-\ln x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3+2x\Delta x+(\Delta x)^2-\Delta x}{\Delta x}\ln x+\left(x^3+x^2-x\right)\lim_{\Delta x\to 0}\frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\left(3x^2+3x\Delta x+(\Delta x)^2+2x+\Delta x-1\right)\ln x+\left(x^3+x^2-x\right)\lim_{\Delta x\to 0}\frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}}\frac{1}{x}$$

$$=\left(3x^2+2x-1\right)\ln x+\left(x^3+x^2-x\right)\frac{1}{x}=\left(3x^2+2x-1\right)\ln x+x^2+x-1$$

$6.(9)\ y=x^3\log_2 x+x^2\log_3 x=\dfrac{x^3\ln x}{\ln 2}+\dfrac{x^2\ln x}{\ln 3}=\left(\dfrac{x^3}{\ln 2}+\dfrac{x^2}{\ln 3}\right)\ln x$

$$y'=\lim_{\Delta x\to 0}\frac{\left(\frac{(x+\Delta x)^3}{\ln 2}+\frac{(x+\Delta x)^2}{\ln 3}\right)-\left(\frac{x^3}{\ln 2}+\frac{x^2}{\ln 3}\right)}{\Delta x}\ln x+\left(\frac{x^3}{\ln 2}+\frac{x^2}{\ln 3}\right)\lim_{\Delta x\to 0}\frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\frac{3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3}{\ln 2}+\frac{2x\Delta x+(\Delta x)^2}{\ln 3}}{\Delta x}\ln x+\left(\frac{x^3}{\ln 2}+\frac{x^2}{\ln 3}\right)\lim_{\Delta x\to 0}\frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\left(\frac{3x^2+3x\Delta x+(\Delta x)^2}{\ln 2}+\frac{2x+\Delta x}{\ln 3}\right)\ln x+\left(\frac{x^3}{\ln 2}+\frac{x^2}{\ln 3}\right)\lim_{\Delta x\to 0}\frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}}\frac{1}{x}$$

$$=\left(\frac{3x^2}{\ln 2}+\frac{2x}{\ln 3}\right)\ln x+\left(\frac{x^3}{\ln 2}+\frac{x^2}{\ln 3}\right)\frac{1}{x}=\left(\frac{3x^2}{\ln 2}+\frac{2x}{\ln 3}\right)\ln x+\frac{x^2}{\ln 2}+\frac{x}{\ln 3}$$

$6.(12)\ y=\frac{\sin x-x\cos x}{\cos x+x\sin x}$

$$y'=\frac{\lim_{\Delta x\to 0}\frac{(\sin(x+\Delta x)-(x+\Delta x)\cos(x+\Delta x))-(\sin x-x\cos x)}{\Delta x}(\cos x+x\sin x)}{(\cos x+x\sin x)^2}$$

$$-\frac{(\sin x-x\cos x)\lim_{\Delta x\to 0}\frac{(\cos(x+\Delta x)+(x+\Delta x)\sin(x+\Delta x))-(\cos x+x\sin x)}{\Delta x}}{(\cos x+x\sin x)^2}$$

$$=\frac{\lim_{\Delta x\to 0}\frac{(\sin(x+\Delta x)-\sin x)-(x(\cos(x+\Delta x)-\cos x)+\Delta x\cos(x+\Delta x))}{\Delta x}(\cos x+x\sin x)}{(\cos x+x\sin x)^2}$$

$$-\frac{(\sin x-x\cos x)\lim_{\Delta x\to 0}\frac{(\cos(x+\Delta x)-\cos x)+(\Delta x\sin(x+\Delta x)+x(\sin(x+\Delta x)-\sin x))}{\Delta x}}{(\cos x+x\sin x)^2}$$

$$=\frac{\lim_{\Delta x\to 0}\frac{2\cos\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}-\left(-2x\sin\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}+\Delta x\cos(x+\Delta x)\right)}{\Delta x}(\cos x+x\sin x)}{(\cos x+x\sin x)^2}$$

$$-\frac{(\sin x-x\cos x)\lim_{\Delta x\to 0}\frac{-2\sin\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}+\left(\Delta x\sin(x+\Delta x)+2x\cos\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}\right)}{\Delta x}}{(\cos x+x\sin x)^2}$$

$$=\frac{\lim_{\Delta x\to 0}\left[\cos\left(x+\frac{\Delta x}{2}\right)+x\sin\left(x+\frac{\Delta x}{2}\right)-\cos(x+\Delta x)\right](\cos x+x\sin x)}{(\cos x+x\sin x)^2}$$

$$-\frac{(\sin x-x\cos x)\lim_{\Delta x\to 0}\left[-\sin\left(x+\frac{\Delta x}{2}\right)+\sin(x+\Delta x)+x\cos\left(x+\frac{\Delta x}{2}\right)\right]}{(\cos x+x\sin x)^2}$$

$$=\frac{x\sin x(\cos x+x\sin x)}{(\cos x+x\sin x)^2}-\frac{(\sin x-x\cos x)x\cos x}{(\cos x+x\sin x)^2}$$

$$=\frac{x\sin x(\cos x+x\sin x)-(\sin x-x\cos x)x\cos x}{(\cos x+x\sin x)^2}$$

$$=\frac{x^2}{(\cos x+x\sin x)^2}$$

$7.(4)\ y=\sin x$

$$y'=\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x}=\lim_{\Delta x\to 0}\frac{2\sin\frac{\Delta x}{2}\cos\left(x+\frac{\Delta x}{2}\right)}{\Delta x}=\cos x$$

$(x_0,f(x_0))$处切线方向:$(1,y')$切线方程：$y'(x-x_0)-(y-\sin x_0)=0$,即$\cos x_0(x-x_0)-(y-\sin x_0)=0$

$(x_0,f(x_0))$处切线方向:$(y',-1)$切线方程:$(x-x_0)+y'(y-\sin x_0)=0$,即$(x-x_0)+\cos x_0(y-\sin x_0)=0$

$$13.(1)\ f(x)=\begin{cases} x^3, x<0 \\ x^2, 0\le x\le 2 \\ \frac{1}{2}x^3-2x+4, x>2 \end{cases}$$

$x<0$时，$f'(x)=\lim\limits_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3}{\Delta x}=\lim\limits_{\Delta x\to 0}3x^2+3x\Delta x+(\Delta x)^2=3x^2$

$\lim\limits_{x\to 0^-}f'(x)=0$

$0\le x\le 2$时$,f'(x)=\lim\limits_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{2x\Delta x+(\Delta x)^2}{\Delta x}=\lim\limits_{\Delta x\to 0}2x+\Delta x=2x$

$\lim\limits_{x\to 0^+}f'(x)=0=\lim\limits_{x\to 0^-}f'(x)\Rightarrow\lim\limits_{x\to 0}f'(x)=\lim\limits_{x\to 0^+}f'(x)=\lim\limits_{x\to 0^-}f'(x)=0$

$\lim\limits_{x\to 2^-}f'(x)=4$

$x>2$时$,f'(x)=\lim\limits_{\Delta x\to 0}\frac{\left(\frac{1}{2}(x+\Delta x)^3-2(x+\Delta x)+4\right)-\left(\frac{1}{2}x^3-2x+4\right)}{\Delta x}$

$=\lim\limits_{\Delta x\to 0}\frac{\frac{1}{2}\left(3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3\right)-2\Delta x}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{1}{2}\left(3x^2+3x\Delta x+(\Delta x)^2\right)-2=\frac{3}{2}x^2-2$

$\lim\limits_{x\to 2^-}f'(x)=\lim\limits_{x\to 2^-}\frac{3}{2}x^2-2=4=\lim\limits_{x\to 2^-}f'(x)\Rightarrow\lim\limits_{x\to 2}f'(x)=\lim\limits_{x\to 2^+}f'(x)=\lim\limits_{x\to 2^-}f'(x)=4$

$$故f'(x)=\begin{cases} 3x^2, x<0 \\ 2x, 0\le x\le 2 \\ \frac{3}{2}x^2-2, x>2 \end{cases}$$

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$1.(2)\ y=2^{\sin^2 x}$

$y'=2^{\sin^2 x}\cdot\ln 2\cdot 2\sin x\cos x=2^{\sin^2 x+1}\sin(2x)\ln 2$

$1.(4)\ y=\log_3^2\left(x^3+2x+1\right)$

$y'=2\log_3\left(x^3+2x+1\right)\frac{1}{\ln 3}\frac{1}{x^3+2x+1}\left(3x^2+2\right)$

$1.(6)\ y=\arctan\frac{x^2}{a}+\operatorname{arccot}\frac{a}{x^2}$

$$y'=\frac{1}{1+\frac{x^4}{a^2}}\frac{2x}{a}-\frac{1}{1+\frac{a^2}{x^4}}\left(-\frac{2a}{x^3}\right)=\frac{2a}{a+\frac{x^4}{a}}+\frac{2a}{x^3+\frac{a^2}{x}}$$

$1.(13)\ y=\sqrt{x+\sqrt{x+\sqrt{x}}}$

$$y'=\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left(x+\sqrt{x+\sqrt{x}}\right)'=\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left(1+\frac{1}{2\sqrt{x+\sqrt{x}}}\left(x+\sqrt{x}\right)'\right)$$

$$=\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left(1+\frac{1+\frac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}\right)=\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left(1+\frac{\frac{2\sqrt{x}+1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}\right)$$

$$=\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left(1+\frac{2\sqrt{x}+1}{4\sqrt{x+\sqrt{x}}\sqrt{x}}\right)=\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\frac{4\sqrt{x+\sqrt{x}}\sqrt{x}+2\sqrt{x}+1}{4\sqrt{x+\sqrt{x}}\sqrt{x}}$$

$$=\frac{4\sqrt{x+\sqrt{x}}\sqrt{x}+2\sqrt{x}+1}{8\sqrt{x+\sqrt{x+\sqrt{x}}}\sqrt{x+\sqrt{x}}\sqrt{x}}$$

$1.(19)\, y=\ln\left(\sqrt{a^2+x^2}+\sqrt{a^2-x^2}\right)$

$$y'=\frac{1}{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}\left(\sqrt{a^2+x^2}+\sqrt{a^2-x^2}\right)'=\frac{1}{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}\left(\frac{x}{\sqrt{a^2+x^2}}-\frac{x}{\sqrt{a^2-x^2}}\right)$$

$$=\frac{x\left(\sqrt{a^2-x^2}-\sqrt{a^2+x^2}\right)}{\left(\sqrt{a^2+x^2}+\sqrt{a^2-x^2}\right)\sqrt{a^2+x^2}\sqrt{a^2-x^2}}=\frac{-2x^3}{\left(\sqrt{a^2+x^2}+\sqrt{a^2-x^2}\right)^2\sqrt{a^2+x^2}\sqrt{a^2-x^2}}$$

$1.(21)\, y=\arctan\frac{x^2-1}{x^2+1}$

$$y'=\frac{1}{1+\left(\frac{x^2-1}{x^2+1}\right)^2}\left(\frac{x^2-1}{x^2+1}\right)'=\frac{1}{1+\left(\frac{x^2-1}{x^2+1}\right)^2}\left(1-\frac{2}{x^2+1}\right)'=\frac{1}{1+\left(\frac{x^2-1}{x^2+1}\right)^2}\frac{4x}{\left(x^2+1\right)^2}$$

$$=\frac{4x}{\left(x^2+1\right)^2+\left(x^2-1\right)^2}=\frac{2x}{x^4+1}$$

$1.(24)\, y=\arctan\left(x+\sqrt{1+x^2}\right)$

$$y'=\frac{1}{1+\left(x+\sqrt{1+x^2}\right)^2}\left(x+\sqrt{1+x^2}\right)'=\frac{1}{1+\left(x+\sqrt{1+x^2}\right)^2}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$=\frac{1}{1+\left(x+\sqrt{1+x^2}\right)^2}\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}=\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}+\sqrt{1+x^2}\left(x+\sqrt{1+x^2}\right)^2}$$

$1.(27)\, y=x\arctan x-\frac{1}{2}\ln\left(1+x^2\right)-\frac{1}{2}\left(\arctan x\right)^2$

$$y=\arctan x+\frac{x}{1+x^2}-\frac{x}{1+x^2}-\frac{\arctan x}{1+x^2}=\frac{x^2\arctan x}{1+x^2}$$

$2.(2)\, y=\frac{1}{2}\arctan\sqrt[4]{1+x^4}-\frac{1}{4}\ln\frac{\sqrt[4]{1+x^4}-1}{\sqrt[4]{1+x^4}+1}$

$\sqrt[4]{1+x^4}\triangleq t$

$$\frac{dy}{dx}=\frac{d\left(\frac{1}{2}\arctan t-\frac{1}{4}\ln\frac{t-1}{t+1}\right)}{dt}\frac{dt}{dx}=\left(\frac{1}{2\left(1+\left(1+x^4\right)^{\frac{1}{2}}\right)}-\frac{1}{4\left(\left(1+x^4\right)^{\frac{1}{2}}-1\right)}\right)\frac{1}{4\left(1+x^4\right)^{\frac{3}{4}}}$$

$2.(4)\, y=\frac{e^{-x^2}\arcsin e^{-x^2}}{\sqrt{1-e^{-2x^2}}}+\frac{1}{2}\ln\left(1-e^{-2x^2}\right)$

$e^{-x^2}\triangleq t, obviously, t\in(0,1)$

$$\frac{dy}{dx}=\frac{d\left(\frac{t\arcsin t}{\sqrt{1-t^2}}+\frac{1}{2}\ln\left(1-t^2\right)\right)}{dt}\frac{dt}{dx}$$

$t\triangleq\sin\alpha,\alpha=\arcsin t$

$$\frac{dy}{dx}=\frac{d\left(\alpha\tan\alpha+\ln\cos\alpha\right)}{d\alpha}\frac{d\alpha}{dt}\frac{dt}{dx}$$

$$=\frac{d\left(\alpha\tan\alpha+\ln\cos\alpha\right)}{d\alpha}\frac{d\alpha}{dt}\frac{dt}{dx}$$

$$=\left(\tan\alpha+\alpha\sec^2\alpha-\tan\alpha\right)\frac{1}{\sqrt{1-t^2}}e^{-x^2}\left(-2x\right)$$

$$=-\alpha\sec^2\alpha\frac{2}{\sqrt{1-t^2}}e^{-x^2}x$$

$$=-\arcsin t\frac{2}{\left(1-t^2\right)^{\frac{3}{2}}}e^{-x^2}x$$

$$=-\arcsin e^{-x^2}\frac{2}{\left(1-e^{-2x^2}\right)^{\frac{3}{2}}}e^{-x^2}x$$

$$3.(1)\,let\ f(x)=\sum_{k=0}^{n}x^k=\frac{1-x^{n+1}}{1-x}$$

$$f'(x)=\left(\sum_{k=0}^{n}x^k\right)'=\sum_{k=0}^{n-1}(k+1)x^k$$

$$f'(x)=\left(\frac{1-x^{n+1}}{1-x}\right)'=\frac{-(n+1)x^n(1-x)+(1-x^{n+1})}{(1-x)^2}=\frac{(n+1)x^n(x-1)+(1-x^{n+1})}{(1-x)^2}=\frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}$$

$$3.(3)\,let\ f(x)=-\sum_{k=1}^{n}\cos kx$$

$$f'(x)=-\left(\sum_{k=1}^{n}\cos kx\right)'=\sum_{k=1}^{n}k\sin kx$$

$$f(x)=-\sum_{k=1}^{n}\cos kx=-\frac{\sum_{k=1}^{n}\sin\frac{x}{2}\cos kx}{\sin\frac{x}{2}}=-\frac{\sum_{k=1}^{n}\left(\sin\left(kx-\frac{x}{2}\right)-\sin\left(kx+\frac{x}{2}\right)\right)}{2\sin\frac{x}{2}}$$

$$=\frac{\sin\left(nx+\frac{x}{2}\right)-\sin\frac{x}{2}}{2\sin\frac{x}{2}}=\frac{\sin\left(nx+\frac{x}{2}\right)}{2\sin\frac{x}{2}}-\frac{1}{2}$$

$$f'(x)=\left(\frac{\sin\left(nx+\frac{x}{2}\right)}{2\sin\frac{x}{2}}-\frac{1}{2}\right)'=\frac{\left(n+\frac{1}{2}\right)\cos\left(n+\frac{1}{2}\right)x\sin\frac{x}{2}-\frac{1}{2}\sin\left(nx+\frac{x}{2}\right)\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}$$

$$\Rightarrow\sum_{k=1}^{n}k\sin kx=\frac{\left(n+\frac{1}{2}\right)\cos\left(n+\frac{1}{2}\right)x\sin\frac{x}{2}-\frac{1}{2}\sin\left(nx+\frac{x}{2}\right)\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}$$

$$3.(5)\,let\ f(x)=\sum_{k=1}^{n}\ln\cos\frac{x}{2^k}$$

$$f'(x)=\left(\sum_{k=1}^{n}\ln\cos\frac{x}{2^k}\right)'=-\sum_{k=1}^{n}\frac{1}{2^k}\tan\frac{x}{2^k}$$

$$f(x)=\sum_{k=1}^{n}\ln\cos\frac{x}{2^k}=\ln\left(\prod_{k=1}^{n}\cos\frac{x}{2^k}\right)=\ln\left(\frac{\sin\frac{x}{2^n}\prod_{k=1}^{n}\cos\frac{x}{2^k}}{\sin\frac{x}{2^n}}\right)=\ln\left(\frac{\sin\frac{x}{2^{n-1}}\prod_{k=1}^{n-1}\cos\frac{x}{2^k}}{2\sin\frac{x}{2^n}}\right)$$

$$=\cdots=\ln\left(\frac{\sin x}{2^n\sin\frac{x}{2^n}}\right)=\ln\sin x-\ln\sin\frac{x}{2^n}-n\ln 2$$

$$f'(x)=\left(\ln\sin x-\ln\sin\frac{x}{2^n}-n\ln 2\right)'=\cot x-\frac{1}{2^n}\cot\frac{x}{2^n}$$

$$\Rightarrow-\sum_{k=1}^{n}\frac{1}{2^k}\tan\frac{x}{2^k}=\cot x-\frac{1}{2^n}\cot\frac{x}{2^n}$$

$$\Rightarrow\sum_{k=1}^{n}\frac{1}{2^k}\tan\frac{x}{2^k}=\frac{1}{2^n}\cot\frac{x}{2^n}-\cot x$$

$$7.(5)\, x^3+y^3-xy=0$$

$$\Rightarrow \frac{d\left(x^3+y^3(x)-xy(x)\right)}{dx}=0 \Rightarrow \frac{dx^3}{dx}+\frac{dy^3(x)}{dx}=\frac{dxy(x)}{dx} \Rightarrow 3x^2+\frac{dy^3}{dy}\frac{dy}{dx}=\frac{dy}{dx}+y\frac{dx}{dx} \Rightarrow 3x^2+3y^2\frac{dy}{dx}=\frac{dy}{dx}+y$$

$$\Rightarrow \left(3y^2-1\right)\frac{dy}{dx}=y-3x^2 \Rightarrow y'(x)=\frac{dy}{dx}=\frac{y-3x^2}{3y^2-1}$$

$$7.(6)\, \arctan\frac{y}{x}=\ln\sqrt{x^2+y^2}$$

$$\Rightarrow \frac{d\arctan\frac{y}{x}}{dx}=\frac{d\ln\sqrt{x^2+y^2}}{dx} \Rightarrow \frac{d\arctan\frac{y}{x}}{d\frac{y}{x}}\frac{d\frac{y}{x}}{dx}=\frac{d\ln\sqrt{x^2+y^2}}{d\sqrt{x^2+y^2}}\frac{d\sqrt{x^2+y^2}}{d\left(x^2+y^2\right)}\frac{d\left(x^2+y^2\right)}{dx}$$

$$\Rightarrow \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\frac{dy}{dx}+y\frac{d\left(\frac{1}{x}\right)}{dx}\right)=\frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}\left(2x+\frac{dy^2}{dy}\frac{dy}{dx}\right) \Rightarrow \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}\right)=\frac{1}{x^2+y^2}\left(x+y\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{x^2}{x^2+y^2}\left(x\frac{dy}{dx}-y\right)=\frac{1}{x^2+y^2}\left(x+y\frac{dy}{dx}\right) \Rightarrow x^3\frac{dy}{dx}-x^2y=x+y\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{x+x^2y}{x^3-y}$$

$$8.\begin{cases} x=\dfrac{at^2}{1+t^2} \\ y=\dfrac{at^3}{1+t^2} \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt}=\dfrac{d\dfrac{at^2}{1+t^2}}{dt}=\dfrac{2at\left(1+t^2\right)-2at^3}{\left(1+t^2\right)^2}=\dfrac{2at}{\left(1+t^2\right)^2} \\ \dfrac{dy}{dt}=\dfrac{d\dfrac{at^3}{1+t^2}}{dt}=\dfrac{3at^2\left(1+t^2\right)-2at^4}{\left(1+t^2\right)^2}=\dfrac{3at^2+at^4}{\left(1+t^2\right)^2} \end{cases}$$

$$k=\frac{dy}{dx}=\frac{\frac{3at^2+at^4}{\left(1+t^2\right)^2}}{\frac{2at}{\left(1+t^2\right)^2}}=\frac{3at^2+at^4}{2at}=\frac{3}{2}t+\frac{1}{2}t^3$$

$$8.(4)\begin{cases} x=a\tan t \\ y=a\cos^2 t \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt}=\dfrac{da\tan t}{dt}=a\sec^2 t \\ \dfrac{dy}{dt}=\dfrac{da\cos^2 t}{dt}=-a\sin(2t) \end{cases}$$

$$k=\frac{dy}{dx}=\frac{-a\sin(2t)}{a\sec^2 t}=-2\sin t\cos^3 t$$

$$8.(6)\begin{cases} x=a\left(\cos t+t\sin t\right) \\ y=a\left(\sin t-t\cos t\right) \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt}=\dfrac{da\left(\cos t+t\sin t\right)}{dt}=at\cos t \\ \dfrac{dy}{dt}=\dfrac{da\left(\sin t-t\cos t\right)}{dt}=at\sin t \end{cases}$$

$$k=\frac{dy}{dx}=\frac{at\sin t}{at\cos t}=\tan t$$

$$9.(2)\ y=\frac{x^2}{2+x^2}\sqrt[3]{\frac{(1+x)^2}{2+x^2}}$$

$$\ln y=\ln\left(\frac{x^2}{2+x^2}\sqrt[3]{\frac{(1+x)^2}{2+x^2}}\right)=2\ln x+\frac{2}{3}\ln(1+x)-\frac{4}{3}\ln\left(2+x^2\right)$$

$$\frac{d\ln y}{dx}=\frac{d\left(2\ln x+\frac{2}{3}\ln(1+x)-\frac{4}{3}\ln\left(2+x^2\right)\right)}{dx}=\frac{2}{x}+\frac{2}{3}\frac{1}{1+x}-\frac{4}{3}\frac{2x}{2+x^2}$$

$$\frac{d\ln y}{dx}=\frac{d\ln y}{dy}\frac{dy}{dx}=\frac{y'}{y}$$

$$\Rightarrow \frac{y'}{y}=\frac{2}{x}+\frac{2}{3}\frac{1}{1+x}-\frac{4}{3}\frac{2x}{2+x^2}$$

$$\Rightarrow y'=\left(\frac{2}{x}+\frac{2}{3}\frac{1}{1+x}-\frac{4}{3}\frac{2x}{2+x^2}\right)y=\left(\frac{2}{x}+\frac{2}{3}\frac{1}{1+x}-\frac{4}{3}\frac{2x}{2+x^2}\right)\frac{x^2}{2+x^2}\sqrt[3]{\frac{(1+x)^2}{2+x^2}}$$

$$9.(4)\ y=\left(1+x^2\right)^{\arctan x}$$

$$\ln y=\arctan x\cdot\ln\left(1+x^2\right)$$

$$\frac{d\ln y}{dx}=\frac{d\left(\arctan x\cdot\ln\left(1+x^2\right)\right)}{dx}=\arctan x\cdot\frac{d\left(\ln\left(1+x^2\right)\right)}{dx}+\frac{d(\arctan x)}{dx}\cdot\ln\left(1+x^2\right)$$

$$=\arctan x\cdot\frac{2x}{1+x^2}+\frac{\ln\left(1+x^2\right)}{1+x^2}$$

$$\frac{d\ln y}{dx}=\frac{d\ln y}{dy}\frac{dy}{dx}=\frac{y'}{y}$$

$$\Rightarrow \frac{y'}{y}=\frac{2x\arctan x+\ln\left(1+x^2\right)}{1+x^2}$$

$$\Rightarrow y'=\frac{2x\arctan x+\ln\left(1+x^2\right)}{1+x^2}y=\frac{2x\arctan x+\ln\left(1+x^2\right)}{1+x^2}\left(1+x^2\right)^{\arctan x}$$

$9.(6)\ y = x^{a^x} + x^{x^a} + x^{x^x}$

$y_1 \triangleq x^{a^x}, y_2 \triangleq x^{x^a}, y_2 \triangleq x^{x^x}$

$y' = y_1' + y_2' + y_3'$

$\bullet \ln y_1 = \ln x^{a^x} = a^x \ln x$

$\ln\ln y_1 = x \ln a \ln\ln x$

$$\frac{d\ln\ln y_1}{dx} = \frac{dx \ln a \ln\ln x}{dx} = \ln a\left(\ln\ln x + x\frac{d\ln\ln x}{dx}\right) = \ln a\left(\ln\ln x + x\frac{\ln\ln x}{d\ln x}\frac{d\ln x}{dx}\right)$$

$$= \ln a\left(\ln\ln x + x\frac{1}{\ln x}\frac{1}{x}\right) = \ln a \ln\ln x + \frac{\ln a}{\ln x}$$

$$\frac{d\ln\ln y_1}{dx} = \frac{d\ln\ln y_1}{d\ln y_1}\frac{d\ln y_1}{dy_1}\frac{dy_1}{dx} = \frac{1}{\ln y_1}\frac{1}{y_1}\frac{dy_1}{dx}$$

$$\Rightarrow \frac{1}{\ln y_1}\frac{1}{y_1}\frac{dy_1}{dx} = \ln a \ln\ln x + \frac{\ln a}{\ln x}$$

$$\Rightarrow y_1' = \frac{dy_1}{dx} = \left(\ln a \ln\ln x + \frac{\ln a}{\ln x}\right) y_1 \ln y_1 = \left(\ln a \ln\ln x + \frac{\ln a}{\ln x}\right) x^{a^x} a^x \ln x$$

$$= \left(\ln a \cdot \ln x \cdot \ln\ln x + \ln a\right) x^{a^x} a^x$$

$\bullet \ln y_2 = \ln x^{x^a} = x^a \ln x$

$$\frac{d\ln y_2}{dx} = \frac{d\left(x^a \ln x\right)}{dx} = x^a \cdot \frac{d(\ln x)}{dx} + \frac{d\left(x^a\right)}{dx} \cdot \ln x = x^{a-1} + a x^{a-1} \cdot \ln x = x^{a-1}\left(1 + a\ln x\right)$$

$$\frac{d\ln y_2}{dx} = \frac{d\ln y_2}{dy_2}\frac{dy_2}{dx} = \frac{y_2'}{y_2}$$

$$\Rightarrow \frac{y_2'}{y_2} = x^{a-1}\left(1 + a\ln x\right)$$

$$\Rightarrow y_2' = x^{a-1}\left(1 + a\ln x\right) y_2 = x^{a-1}\left(1 + a\ln x\right) x^{x^a} = \left(1 + a\ln x\right) x^{x^a + a - 1}$$

$\bullet \ln y_3 = \ln x^{x^x} = x^x \ln x$

$\ln\ln y_3 = x \ln x \ln\ln x$

$$\frac{d\ln\ln y_3}{dx} = \frac{dx \ln x \ln\ln x}{dx} = \frac{dx}{dx}\ln x \ln\ln x + \frac{d\ln x}{dx} x \ln\ln x + \frac{d\ln\ln x}{dx} x \ln x$$

$$= \ln x \ln\ln x + \ln\ln x + \frac{d\ln\ln x}{d\ln x}\frac{d\ln x}{dx} x \ln x = \ln x \ln\ln x + \ln\ln x + 1$$

$$\frac{d\ln\ln y_3}{dx} = \frac{d\ln\ln y_3}{d\ln y_3}\frac{d\ln y_3}{dy_3}\frac{dy_3}{dx} = \frac{1}{\ln y_3}\frac{1}{y_3}\frac{dy_3}{dx}$$

$$\Rightarrow \frac{1}{\ln y_3}\frac{1}{y_3}\frac{dy_3}{dx} = \ln x \ln\ln x + \ln\ln x + 1$$

$$\Rightarrow y_3' = \frac{dy_3}{dx} = \left(\ln x \ln\ln x + \ln\ln x + 1\right) y_3 \ln y_3 = \left(\ln x \ln\ln x + \ln\ln x + 1\right) x^{x^x} x^x \ln x$$

$$= \left(\ln x \ln\ln x + \ln\ln x + 1\right) x^{x^x + x} \ln x$$

$\bullet$ *Hence*, $y' = y_1' + y_2' + y_3'$

$$= \left(\ln a \cdot \ln x \cdot \ln\ln x + \ln a\right) x^{a^x} a^x + \left(1 + a\ln x\right) x^{x^a + a - 1} + \left(\ln x \ln\ln x + \ln\ln x + 1\right) x^{x^x + x} \ln x$$

$$9.(8)\, y=(\sin x)^{\cos x}(\cos x)^{\sin x}$$

$$\ln y=\ln\left((\sin x)^{\cos x}(\cos x)^{\sin x}\right)=\cos x\ln\sin x+\sin x\ln\cos x$$

$$\frac{y'}{y}=\frac{d\ln y}{dx}=\frac{d(\cos x\ln\sin x+\sin x\ln\cos x)}{dx}=\frac{d\cos x\ln\sin x}{dx}+\frac{d\sin x\ln\cos x}{dx}$$

$$=\cos x\frac{d\ln\sin x}{dx}+\frac{d\cos x}{dx}\ln\sin x+\sin x\frac{d\ln\cos x}{dx}+\frac{d\sin x}{dx}\ln\cos x$$

$$=\cos x\frac{d\ln\sin x}{d\sin x}\frac{d\sin x}{dx}+\frac{d\cos x}{dx}\ln\sin x+\sin x\frac{d\ln\cos x}{d\cos x}\frac{d\cos x}{dx}+\frac{d\sin x}{dx}\ln\cos x$$

$$=\cos x\frac{1}{\sin x}\cos x-\sin x\ln\sin x+\sin x\frac{1}{\cos x}(-\sin x)+\cos x\ln\cos x$$

$$=\frac{\cos^2 x}{\sin x}-\sin x\ln\sin x-\frac{\sin^2 x}{\cos x}+\cos x\ln\cos x=\frac{\cos^3 x-\sin^3 x}{\sin x\cos x}-\sin x\ln\sin x+\cos x\ln\cos x$$

$$=\frac{(\cos x-\sin x)(\cos^2 x+\cos x\sin x+\sin^2 x)}{\sin x\cos x}-\sin x\ln\sin x+\cos x\ln\cos x$$

$$=\frac{(\cos x-\sin x)(1+\cos x\sin x)}{\sin x\cos x}-\sin x\ln\sin x+\cos x\ln\cos x$$

$$\Rightarrow y'=\left(\frac{(\cos x-\sin x)(1+\cos x\sin x)}{\sin x\cos x}-\sin x\ln\sin x+\cos x\ln\cos x\right)y$$

$$=\left(\frac{(\cos x-\sin x)(1+\cos x\sin x)}{\sin x\cos x}-\sin x\ln\sin x+\cos x\ln\cos x\right)(\sin x)^{\cos x}(\cos x)^{\sin x}$$

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$$1.\Delta f(1)=f(1+\Delta x)-f(1)=\left((1+\Delta x)^3-3(1+\Delta x)+2\right)-\left(1^3-3+2\right)$$

$$=1+3\Delta x+3(\Delta x)^2+(\Delta x)^3-3(1+\Delta x)+2=3(\Delta x)^2+(\Delta x)^3$$

$$df(1)=\left(d\left(x^3-3x+2\right)\right)|_{x=1}=\left(3x^2dx-3dx\right)|_{x=1}=0$$

$$\Delta x=0.1\Rightarrow\Delta f(1)=\left(3(\Delta x)^2+(\Delta x)^3\right)|_{\Delta x=0.1}=0.031$$

$$\Delta x=0.01\Rightarrow\Delta f(1)=\left(3(\Delta x)^2+(\Delta x)^3\right)|_{\Delta x=0.01}=0.000301$$

2.(2) $y=\arctan\left(x+\sqrt{1+x^2}\right)$

$$\frac{\Delta y}{\Delta x}=\frac{\Delta\arctan\left(x+\sqrt{1+x^2}\right)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\arctan\left(x+\Delta x+\sqrt{1+(x+\Delta x)^2}\right)-\arctan\left(x+\sqrt{1+x^2}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\arctan\left(\frac{\left(x+\Delta x+\sqrt{1+(x+\Delta x)^2}\right)-\left(x+\sqrt{1+x^2}\right)}{1+\left(x+\Delta x+\sqrt{1+(x+\Delta x)^2}\right)\left(x+\sqrt{1+x^2}\right)}\right)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\arctan\left(\frac{\Delta x+\left(\sqrt{1+(x+\Delta x)^2}-\sqrt{1+x^2}\right)}{1+\left(x+\Delta x+\sqrt{1+(x+\Delta x)^2}\right)\left(x+\sqrt{1+x^2}\right)}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\arctan\left(\frac{\Delta x+\frac{2x\Delta x+(\Delta x)^2}{\sqrt{1+(x+\Delta x)^2}+\sqrt{1+x^2}}}{1+\left(x+\Delta x+\sqrt{1+(x+\Delta x)^2}\right)\left(x+\sqrt{1+x^2}\right)}\right)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x+\frac{2x\Delta x+(\Delta x)^2}{\sqrt{1+(x+\Delta x)^2}+\sqrt{1+x^2}}}{1+\left(x+\Delta x+\sqrt{1+(x+\Delta x)^2}\right)\left(x+\sqrt{1+x^2}\right)}\frac{1}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{1+\frac{2x+\Delta x}{\sqrt{1+(x+\Delta x)^2}+\sqrt{1+x^2}}}{1+\left(x+\Delta x+\sqrt{1+(x+\Delta x)^2}\right)\left(x+\sqrt{1+x^2}\right)}=\frac{1+\lim\limits_{\Delta x\to 0}\frac{2x+\Delta x}{\sqrt{1+(x+\Delta x)^2}+\sqrt{1+x^2}}}{1+\lim\limits_{\Delta x\to 0}\left(x+\Delta x+\sqrt{1+(x+\Delta x)^2}\right)\left(x+\sqrt{1+x^2}\right)}=\frac{1+\frac{x}{\sqrt{1+x^2}}}{1+\left(x+\sqrt{1+x^2}\right)^2}$$

2.(4) $y=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)$

$$\frac{\Delta y}{\Delta x}=\frac{\Delta\left(\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)\right)}{\Delta x}=\frac{\Delta\left(\frac{x}{2}\sqrt{x^2+a^2}\right)}{\Delta x}+\frac{\Delta\left(\frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)\right)}{\Delta x}=\frac{\Delta\left(\frac{x}{2}\sqrt{x^2+a^2}\right)}{\Delta x}+\frac{\frac{a^2}{2}\Delta\left(\ln\left(x+\sqrt{x^2+a^2}\right)\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\left(\frac{(x+\Delta x)}{2}\sqrt{(x+\Delta x)^2+a^2}\right)-\left(\frac{x}{2}\sqrt{x^2+a^2}\right)}{\Delta x}+\frac{a^2}{2}\lim_{\Delta x\to 0}\frac{\ln\left((x+\Delta x)+\sqrt{(x+\Delta x)^2+a^2}\right)-\ln\left(x+\sqrt{x^2+a^2}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\frac{\Delta x}{2}\sqrt{(x+\Delta x)^2+a^2}+\frac{x}{2}\left(\sqrt{(x+\Delta x)^2+a^2}-\sqrt{x^2+a^2}\right)}{\Delta x}+\frac{a^2}{2}\lim_{\Delta x\to 0}\frac{\ln\left(\frac{(x+\Delta x)+\sqrt{(x+\Delta x)^2+a^2}}{x+\sqrt{x^2+a^2}}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\frac{\Delta x}{2}\sqrt{(x+\Delta x)^2+a^2}+\frac{x}{2}\frac{2x\Delta x+(\Delta x)^2}{\sqrt{(x+\Delta x)^2+a^2}+\sqrt{x^2+a^2}}}{\Delta x}+\frac{a^2}{2}\lim_{\Delta x\to 0}\frac{\ln\left(1+\frac{\Delta x+\sqrt{(x+\Delta x)^2+a^2}-\sqrt{x^2+a^2}}{x+\sqrt{x^2+a^2}}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{1}{2}\sqrt{(x+\Delta x)^2+a^2}+\frac{x}{2}\frac{2x+\Delta x}{\sqrt{(x+\Delta x)^2+a^2}+\sqrt{x^2+a^2}}+\frac{a^2}{2}\lim_{\Delta x\to 0}\frac{\frac{\Delta x+\sqrt{(x+\Delta x)^2+a^2}-\sqrt{x^2+a^2}}{x+\sqrt{x^2+a^2}}}{\Delta x}$$

$$=\frac{1}{2}\sqrt{x^2+a^2}+\frac{x^2}{2\sqrt{x^2+a^2}}+\frac{a^2}{2}\lim_{\Delta x\to 0}\frac{\Delta x+\frac{2x\Delta x+(\Delta x)^2}{\sqrt{(x+\Delta x)^2+a^2}+\sqrt{x^2+a^2}}}{\Delta x\left(x+\sqrt{x^2+a^2}\right)}$$

$$=\frac{1}{2}\sqrt{x^2+a^2}+\frac{x^2}{2\sqrt{x^2+a^2}}+\frac{a^2}{2}\lim_{\Delta x\to 0}\frac{1+\frac{2x+\Delta x}{\sqrt{(x+\Delta x)^2+a^2}+\sqrt{x^2+a^2}}}{\left(x+\sqrt{x^2+a^2}\right)}=\frac{1}{2}\sqrt{x^2+a^2}+\frac{x^2}{2\sqrt{x^2+a^2}}+\frac{a^2}{2}\frac{1+\frac{x}{\sqrt{x^2+a^2}}}{\left(x+\sqrt{x^2+a^2}\right)}$$

$$=\frac{1}{2}\sqrt{x^2+a^2}+\frac{x^2}{2\sqrt{x^2+a^2}}+\frac{a^2}{2}\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}\left(x+\sqrt{x^2+a^2}\right)}=\frac{1}{2}\sqrt{x^2+a^2}+\frac{x^2+a^2}{2\sqrt{x^2+a^2}}=\sqrt{x^2+a^2}$$

2.(6) $y=\arccos\frac{x}{\sqrt{2}}+\frac{x^2}{2}\arcsin\frac{x}{\sqrt{2}}$

$$\frac{\Delta y}{\Delta x}=\frac{\Delta\left(\arccos\frac{x}{\sqrt{2}}+\frac{x^2}{2}\arcsin\frac{x}{\sqrt{2}}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\left(\arccos\frac{x+\Delta x}{\sqrt{2}}+\frac{(x+\Delta x)^2}{2}\arcsin\frac{x+\Delta x}{\sqrt{2}}\right)-\left(\arccos\frac{x}{\sqrt{2}}+\frac{x^2}{2}\arcsin\frac{x}{\sqrt{2}}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\left(\arccos\frac{x+\Delta x}{\sqrt{2}}-\arccos\frac{x}{\sqrt{2}}\right)+\left(\frac{(x+\Delta x)^2}{2}\arcsin\frac{x+\Delta x}{\sqrt{2}}-\frac{x^2}{2}\arcsin\frac{x}{\sqrt{2}}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\arccos\frac{x+\Delta x}{\sqrt{2}}-\arccos\frac{x}{\sqrt{2}}}{\Delta x}+\lim_{\Delta x\to 0}\frac{\frac{(x+\Delta x)^2}{2}\arcsin\frac{x+\Delta x}{\sqrt{2}}-\frac{x^2}{2}\arcsin\frac{x}{\sqrt{2}}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\arcsin\left(\frac{x}{\sqrt{2}}\sqrt{1-\left(\frac{x+\Delta x}{\sqrt{2}}\right)^2}-\frac{x+\Delta x}{\sqrt{2}}\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}\right)}{\Delta x}+\lim_{\Delta x\to 0}\frac{\left(x\Delta x+\frac{(\Delta x)^2}{2}\right)\arcsin\frac{x+\Delta x}{\sqrt{2}}+\frac{x^2}{2}\left(\arcsin\frac{x+\Delta x}{\sqrt{2}}-\arcsin\frac{x}{\sqrt{2}}\right)}{\Delta x}$$

$$=\frac{1}{\sqrt{2}}\lim_{\Delta x\to 0}\frac{x\left(\sqrt{1-\left(\frac{x+\Delta x}{\sqrt{2}}\right)^2}-\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}\right)-\Delta x\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}{\Delta x}+\lim_{\Delta x\to 0}\left[\left(x+\frac{\Delta x}{2}\right)\arcsin\frac{x+\Delta x}{\sqrt{2}}\right]+\frac{1}{2}\lim_{\Delta x\to 0}\frac{x^2\left(\arcsin\frac{x+\Delta x}{\sqrt{2}}-\arcsin\frac{x}{\sqrt{2}}\right)}{\Delta x}$$

$$=\frac{1}{\sqrt{2}}\lim_{\Delta x\to 0}\frac{x\frac{\left(\frac{x}{\sqrt{2}}\right)^2-\left(\frac{x+\Delta x}{\sqrt{2}}\right)^2}{\sqrt{1-\left(\frac{x+\Delta x}{\sqrt{2}}\right)^2}+\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}-\Delta x\sqrt{1-\frac{x^2}{2}}}{\Delta x}+x\arcsin\frac{x}{\sqrt{2}}+\frac{x^2}{2}\lim_{\Delta x\to 0}\frac{\arcsin\left(\frac{x+\Delta x}{\sqrt{2}}\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}-\frac{x}{\sqrt{2}}\sqrt{1-\left(\frac{x+\Delta x}{\sqrt{2}}\right)^2}\right)}{\Delta x}$$

$$=\frac{1}{\sqrt{2}}\lim_{\Delta x\to 0}\frac{x\frac{\frac{2x\Delta x+(\Delta x)^2}{2}}{\sqrt{1-\left(\frac{x+\Delta x}{\sqrt{2}}\right)^2}+\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}-\Delta x\sqrt{1-\frac{x^2}{2}}}{\Delta x}+x\arcsin\frac{x}{\sqrt{2}}+\frac{x^2}{2}\lim_{\Delta x\to 0}\frac{\frac{x}{\sqrt{2}}\left(\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}-\sqrt{1-\left(\frac{x+\Delta x}{\sqrt{2}}\right)^2}\right)+\frac{\Delta x}{\sqrt{2}}\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}{\Delta x}$$

$$=\frac{1}{\sqrt{2}}\lim_{\Delta x\to 0}\left[x\frac{x+\frac{\Delta x}{2}}{\sqrt{1-\left(\frac{x+\Delta x}{\sqrt{2}}\right)^2}+\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}-\sqrt{1-\frac{x^2}{2}}\right]+x\arcsin\frac{x}{\sqrt{2}}+\frac{x^2}{2}\lim_{\Delta x\to 0}\frac{\frac{x}{\sqrt{2}}\frac{\left(\frac{x+\Delta x}{\sqrt{2}}\right)^2-\left(\frac{x}{\sqrt{2}}\right)^2}{\sqrt{1-\left(\frac{x+\Delta x}{\sqrt{2}}\right)^2}+\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}+\frac{\Delta x}{\sqrt{2}}\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}{\Delta x}$$

$$=\frac{1}{\sqrt{2}}\left[\frac{x^2}{2\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}-\sqrt{1-\frac{x^2}{2}}\right]+x\arcsin\frac{x}{\sqrt{2}}+\frac{x^2}{2}\lim_{\Delta x\to 0}\frac{\frac{x}{\sqrt{2}}\frac{\frac{2x\Delta x+(\Delta x)^2}{2}}{\sqrt{1-\left(\frac{x+\Delta x}{\sqrt{2}}\right)^2}+\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}+\frac{\Delta x}{\sqrt{2}}\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}{\Delta x}$$

$$=\frac{1}{\sqrt{2}}\left[\frac{x^2}{2\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}-\sqrt{1-\frac{x^2}{2}}\right]+x\arcsin\frac{x}{\sqrt{2}}+\frac{x^2}{2}\lim_{\Delta x\to 0}\left[\frac{x}{\sqrt{2}}\frac{x+\frac{\Delta x}{2}}{\sqrt{1-\left(\frac{x+\Delta x}{\sqrt{2}}\right)^2}+\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}+\frac{1}{\sqrt{2}}\sqrt{1-\frac{x^2}{2}}\right]$$

$$=\frac{1}{\sqrt{2}}\left[\frac{x^2-2\left(1-\frac{x^2}{2}\right)}{2\sqrt{1-\frac{x^2}{2}}}\right]+x\arcsin\frac{x}{\sqrt{2}}+\frac{x^2}{2}\left[\frac{x}{\sqrt{2}}\frac{x}{2\sqrt{1-\frac{x^2}{2}}}+\frac{1}{\sqrt{2}}\frac{1-\frac{x^2}{2}}{\sqrt{1-\frac{x^2}{2}}}\right]$$

$$=-\frac{1}{\sqrt{2}}\frac{1}{\sqrt{1-\frac{x^2}{2}}}+x\arcsin\frac{x}{\sqrt{2}}+\frac{x^2}{2}\left[\frac{1}{\sqrt{2}}\frac{x^2}{2\sqrt{1-\frac{x^2}{2}}}+\frac{1}{\sqrt{2}}\frac{2-x^2}{2\sqrt{1-\frac{x^2}{2}}}\right]=-\frac{1}{\sqrt{2-x^2}}+x\arcsin\frac{x}{\sqrt{2}}+\frac{x^2}{2}\frac{1}{\sqrt{2}}\frac{2}{2\sqrt{1-\frac{x^2}{2}}}$$

$$=-\frac{1}{\sqrt{2-x^2}}+x\arcsin\frac{x}{\sqrt{2}}+\frac{x^2}{2}\frac{1}{\sqrt{2-x^2}}=x\arcsin\frac{x}{\sqrt{2}}+\frac{x^2-2}{2}\frac{1}{\sqrt{2-x^2}}=x\arcsin\frac{x}{\sqrt{2}}-\frac{\sqrt{2-x^2}}{2}$$

$$2.(8)\ y=-\frac{\cos x}{2\sin^2 x}+\ln\sqrt{\frac{1+\cos x}{\sin x}}$$

$$\frac{\Delta y}{\Delta x}=\frac{\Delta\left(-\frac{\cos x}{2\sin^2 x}+\ln\sqrt{\frac{1+\cos x}{\sin x}}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\left(-\frac{\cos(x+\Delta x)}{2\sin^2(x+\Delta x)}+\ln\sqrt{\frac{1+\cos(x+\Delta x)}{\sin(x+\Delta x)}}\right)-\left(-\frac{\cos x}{2\sin^2 x}+\ln\sqrt{\frac{1+\cos x}{\sin x}}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\frac{\cos x}{2\sin^2 x}-\frac{\cos(x+\Delta x)}{2\sin^2(x+\Delta x)}}{\Delta x}+\lim_{\Delta x\to 0}\frac{\ln\sqrt{\frac{1+\cos(x+\Delta x)}{\sin(x+\Delta x)}}-\ln\sqrt{\frac{1+\cos x}{\sin x}}}{\Delta x}$$

$$=\frac{1}{2}\lim_{\Delta x\to 0}\frac{\sin^2(x+\Delta x)\cos x-\sin^2 x\cos(x+\Delta x)}{\Delta x\sin^2 x\sin^2(x+\Delta x)}+\frac{1}{2}\lim_{\Delta x\to 0}\frac{\ln\frac{1+\cos(x+\Delta x)}{1+\cos x}\frac{\sin x}{\sin(x+\Delta x)}}{\Delta x}$$

$$=\frac{1}{2}\lim_{\Delta x\to 0}\frac{(\sin x\cos\Delta x+\cos x\sin\Delta x)^2\cos x-\sin^2 x(\cos x\cos\Delta x-\sin x\sin\Delta x)}{\Delta x\sin^2 x\sin^2(x+\Delta x)}+\frac{1}{2}\lim_{\Delta x\to 0}\frac{\ln\frac{(1+\cos x\cos\Delta x-\sin x\sin\Delta x)\sin x}{(1+\cos x)(\sin x\cos\Delta x+\cos x\sin\Delta x)}}{\Delta x}$$

$$=\frac{1}{2}\lim_{\Delta x\to 0}\frac{(\sin^2 x\cos^2\Delta x+2\sin x\cos x\sin\Delta x\cos\Delta x+\cos^2 x\sin^2\Delta x)\cos x-\sin^2 x(\cos x\cos\Delta x-\sin x\sin\Delta x)}{\Delta x\sin^2 x\sin^2(x+\Delta x)}$$

$$\frac{1}{2}\lim_{\Delta x\to 0}\frac{\frac{(1+\cos x\cos\Delta x-\sin x\sin\Delta x)\sin x}{(1+\cos x)(\sin x\cos\Delta x+\cos x\sin\Delta x)}-1}{\Delta x}$$

$$=\frac{1}{2}\lim_{\Delta x\to 0}\frac{\sin^2 x\cos x\cos^2\Delta x+2\sin x\cos^2 x\sin\Delta x\cos\Delta x+\cos^3 x\sin^2\Delta x-\sin^2 x\cos x\cos\Delta x+\sin^3 x\sin\Delta x}{\Delta x\sin^2 x\sin^2(x+\Delta x)}$$

$$+\frac{1}{2}\lim_{\Delta x\to 0}\frac{(1+\cos x\cos\Delta x-\sin x\sin\Delta x)\sin x-(1+\cos x)(\sin x\cos\Delta x+\cos x\sin\Delta x)}{\Delta x(1+\cos x)(\sin x\cos\Delta x+\cos x\sin\Delta x)}$$

$$=\frac{1}{2}\lim_{\Delta x\to 0}\frac{\sin^2 x\cos x\cos\Delta x(\cos\Delta x-1)+2\sin x\cos^2 x\sin\Delta x\cos\Delta x+\cos^3 x\sin^2\Delta x+\sin^3 x\sin\Delta x}{\Delta x\sin^2 x\sin^2(x+\Delta x)}$$

$$+\frac{1}{2}\lim_{\Delta x\to 0}\frac{\sin x+\sin x\cos x\cos\Delta x-\sin^2 x\sin\Delta x-\sin x\cos\Delta x-\cos x\sin\Delta x-\sin x\cos x\cos\Delta x-\cos^2 x\sin\Delta x}{\Delta x(1+\cos x)(\sin x\cos\Delta x+\cos x\sin\Delta x)}$$

$$=\frac{1}{2}\frac{1}{\sin^4 x}\lim_{\Delta x\to 0}2\sin x\cos^2 x\cos\Delta x+\cos^3 x\sin\Delta x+\sin^3 x+\frac{1}{2}\lim_{\Delta x\to 0}\frac{\sin x(1-\cos\Delta x)-\sin^2 x\sin\Delta x-\cos x\sin\Delta x-\cos^2 x\sin\Delta x}{\Delta x(1+\cos x)(\sin x\cos\Delta x+\cos x\sin\Delta x)}$$

$$=\frac{1}{2}\frac{1}{\sin^4 x}\lim_{\Delta x\to 0}\left(2\sin x\cos^2 x\cos\Delta x+\cos^3 x\sin\Delta x+\sin^3 x\right)+\frac{1}{2}\frac{1}{(1+\cos x)\sin x}\left(-\sin^2 x-\cos x-\cos^2 x\right)$$

$$=\frac{2\sin x\cos^2 x+\sin^3 x}{2\sin^4 x}-\frac{1}{2\sin x}=\frac{\cos^2 x}{\sin^3 x}=\frac{1-\sin^2 x}{\sin^3 x}=\frac{1}{\sin^3 x}-\frac{1}{\sin x}$$

$$3.(1)\, dy = d\sqrt{u^2+v^2} = \frac{d\sqrt{u^2+v^2}}{d\left(u^2+v^2\right)} d\left(u^2+v^2\right) = \frac{1}{2\sqrt{u^2+v^2}} d\left(u^2+v^2\right)$$

$$= \frac{du^2+dv^2}{2\sqrt{u^2+v^2}} = \frac{\frac{du^2}{du}du + \frac{dv^2}{dv}dv}{2\sqrt{u^2+v^2}} = \frac{udu+vdv}{\sqrt{u^2+v^2}}$$

$$3.(3)\, dy = d\ln\sqrt{u^2+v^2} = \frac{d\ln\sqrt{u^2+v^2}}{d\sqrt{u^2+v^2}} d\sqrt{u^2+v^2} = \frac{1}{\sqrt{u^2+v^2}} d\sqrt{u^2+v^2}$$

$$= \frac{1}{\sqrt{u^2+v^2}} \frac{udu+vdv}{\sqrt{u^2+v^2}} = \frac{udu+vdv}{u^2+v^2}$$

4.*proof* :

$$(g\circ f)'(x_0) = \lim_{\Delta x\to 0} \frac{(g\circ f)(x_0+\Delta x)-(g\circ f)(x_0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{(g\circ f)(x_0+\Delta x)-(g\circ f)(x_0)}{f(x_0+\Delta x)-f(x_0)} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{g\left(f(x_0+\Delta x)\right)-g\left(f(x_0)\right)}{f(x_0+\Delta x)-f(x_0)} \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$$= g'\left(f(x_0)\right) f'(x_0) = g'(y_0) f'(x_0)$$

6.

$$af(x_0+\Delta x)+bf(x_0+2\Delta x)-f(x_0) = o(\Delta x)\left(when\ \Delta x\to 0\right)$$

$$\frac{af(x_0+\Delta x)+bf(x_0+2\Delta x)-f(x_0)}{\Delta x} = \frac{o(\Delta x)}{\Delta x} = 0\left(when\ \Delta x\to 0\right)$$

$$\left(when\ \Delta x\to 0\right)\ 0 = \frac{af(x_0+\Delta x)+bf(x_0+2\Delta x)-f(x_0)}{\Delta x}$$

$$= \frac{a\left(f(x_0+\Delta x)-f(x_0)\right)+b\left(f(x_0+2\Delta x)-f(x_0)\right)-(1-a-b)f(x_0)}{\Delta x}$$

$$= af'(x_0)+2bf'(x_0)+(a+b-1)\frac{f(x_0)}{\Delta x}$$

$$= (a+2b)f'(x_0)+(a+b-1)\frac{f(x_0)}{\Delta x}$$

$$\Rightarrow \begin{cases} a+2b=0 \\ a+b-1=0 \end{cases}$$

$$\Rightarrow \begin{cases} a=2 \\ b=-1 \end{cases}$$

$10.proof:$

$$f_+'(0)=\lim_{\Delta x\to 0^+}\frac{f(0+\Delta x)-f(0)}{\Delta x}=\lim_{\Delta x\to 0^+}\frac{f(\Delta x)}{\Delta x}=\lim_{x\to 0^+}\frac{f(x)}{x}\overset{Heine\ Thm}{=}\lim_{n\to\infty}\frac{f(x_n)}{x_n}\left(\lim_{n\to\infty}x_n=0^+\right)$$

$\forall\varepsilon>0,$

$\exists N_1\in\mathbb{N},s.t.\forall n>N_1,$

$$\left|\frac{f(x_n)}{x_n}-f_+'(0)\right|<\varepsilon,$$

$\exists N_2\in\mathbb{N},s.t.\forall n>N_2,$

$x_n>0.$

$then\ for\ \forall n>\max\{N_1,N_2\},$

$$f_+'(0)-\varepsilon<\frac{f(x_n)}{x_n}<f_+'(0)+\varepsilon$$

$$\Rightarrow f_+'(0)x_n-\varepsilon x_n<f(x_n)<f_+'(0)x_n+\varepsilon x_n$$

$choose\ x_{ni}=\dfrac{i}{n^2},i=1,2,\cdots,n$

$\Rightarrow\forall i\in\{1,2,\cdots,n\},$

$$f_+'(0)x_{ni}-\varepsilon x_{ni}<f(x_{ni})<f_+'(0)x_{ni}+\varepsilon x_{ni}$$

$$\Rightarrow\sum_{i=1}^{n}\left(f_+'(0)x_{ni}-\varepsilon x_{ni}\right)<\sum_{i=1}^{n}f(x_{ni})<\sum_{i=1}^{n}\left(f_+'(0)x_{ni}+\varepsilon x_{ni}\right)$$

$$\Rightarrow\sum_{i=1}^{n}\left(f_+'(0)\frac{i}{n^2}-\varepsilon\frac{i}{n^2}\right)<\sum_{i=1}^{n}f\left(\frac{i}{n^2}\right)<\sum_{i=1}^{n}\left(f_+'(0)\frac{i}{n^2}+\varepsilon\frac{i}{n^2}\right)$$

$$\Rightarrow f_+'(0)\frac{n(n+1)}{2n^2}-\varepsilon\frac{n(n+1)}{2n^2}<\sum_{i=1}^{n}f\left(\frac{i}{n^2}\right)<f_+'(0)\frac{n(n+1)}{2n^2}-\varepsilon\frac{n(n+1)}{2n^2}$$

$$\overset{n\to\infty}{\Rightarrow}\frac{f_+'(0)}{2}-\frac{\varepsilon}{2}<\sum_{i=1}^{n}f\left(\frac{i}{n^2}\right)<\frac{f_+'(0)}{2}+\frac{\varepsilon}{2}$$

$\Rightarrow\forall\varepsilon>0,choose\ N=\max\{N_1,N_2\},s.t.\forall n>N,$

$$\left|\sum_{i=1}^{n}f\left(\frac{i}{n^2}\right)-\frac{f_+'(0)}{2}\right|<\frac{\varepsilon}{2}.$$

$$\Rightarrow\lim_{n\to\infty}\sum_{i=1}^{n}f\left(\frac{i}{n^2}\right)=\frac{f_+'(0)}{2}$$

$(1)let\ f(x)=\sin x,then\ f'(x)=\cos x.$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sin\frac{i}{n^2}=\lim_{n\to\infty}\sum_{i=1}^{n}f\left(\frac{i}{n^2}\right)=\frac{f_+'(0)}{2}=\frac{1}{2}$$

$(2)let\ f(x)=\ln(1+x),then\ f'(x)=\dfrac{1}{1+x}.$

$$\lim_{n\to\infty}\prod_{i=1}^{n}\left(1+\frac{i}{n^2}\right)=\lim_{n\to\infty}e^{\sum_{i=1}^{n}\ln\left(1+\frac{i}{n^2}\right)}=e^{\lim_{n\to\infty}\sum_{i=1}^{n}\ln\left(1+\frac{i}{n^2}\right)}=e^{\lim_{n\to\infty}\sum_{i=1}^{n}f\left(\frac{i}{n^2}\right)}=e^{\frac{f_+'(0)}{2}}=\sqrt{e}$$

$(3)let\ f(x)=\ln\cos x,then\ f'(x)=-\tan x.$

$$\lim_{n\to\infty}\prod_{i=1}^{n}\cos\frac{i}{n^2}=\lim_{n\to\infty}e^{\sum_{i=1}^{n}\ln\cos\frac{i}{n^2}}=e^{\lim_{n\to\infty}\sum_{i=1}^{n}\ln\cos\frac{i}{n^2}}=e^{\lim_{n\to\infty}\sum_{i=1}^{n}f\left(\frac{i}{n^2}\right)}=e^{\frac{f_+'(0)}{2}}=e^0=1$$

$11.(1)\sin 29^\circ$

$f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x$

$choose\ f(x)=\sin x,\ x_0=30^\circ,\Delta x=1^\circ, then\ f'(x)=\cos x$

$\sin 29^\circ = f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x=\sin 30^\circ+\cos 30^\circ\cdot 1^\circ=\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\pi}{180}=\frac{180+\sqrt{3}\pi}{360}$

$11.(4)\lg 11$

$f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x$

$choose\ f(x)=\lg x,\ x_0=10,\Delta x=1, then\ f'(x)=\frac{1}{x\ln 10}$

$\lg 11=f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x=\lg 10+\frac{1}{10\ln 10}\cdot 1=1+\frac{1}{10\ln 10}$