15.设$\alpha_1,\alpha_2,\cdots,\alpha_s$是欧氏空间$V$内一个向量组,令

$$D=\begin{bmatrix}(\alpha_1,\alpha_1) & (\alpha_1,\alpha_2) & \cdots & (\alpha_1,\alpha_s)\\ (\alpha_2,\alpha_1) & (\alpha_2,\alpha_2) & \cdots & (\alpha_2,\alpha_s)\\ \vdots & \vdots & & \vdots\\ (\alpha_s,\alpha_1) & (\alpha_s,\alpha_2) & \cdots & (\alpha_s,\alpha_s)\end{bmatrix}$$

证明：$\alpha_1,\alpha_2,\cdots,\alpha_s$线性无关的充分必要条件是$\det(D)\neq 0$.

Pf: ①若$\alpha_1,\alpha_2,\cdots,\alpha_s$线性无关,则'$\sum c_i\alpha_i=0\Rightarrow c_i=0$'......$(\star)$

考虑D的列向量组$\begin{bmatrix}(\alpha_1,\alpha_i)\\(\alpha_2,\alpha_i)\\\vdots\\(\alpha_s,\alpha_i)\end{bmatrix}$,令$\sum c_i\begin{bmatrix}(\alpha_1,\alpha_i)\\(\alpha_2,\alpha_i)\\\vdots\\(\alpha_s,\alpha_i)\end{bmatrix}=0$,则$\begin{bmatrix}\left(\alpha_1,\sum c_i\alpha_i\right)\\\left(\alpha_2,\sum c_i\alpha_i\right)\\\vdots\\\left(\alpha_s,\sum c_i\alpha_i\right)\end{bmatrix}=0$

于是$\left(\alpha_j,\sum c_i\alpha_i\right)=0$,于是$\sum c_j\left(\alpha_j,\sum c_i\alpha_i\right)=0$,即$\left(\sum c_i\alpha_i,\sum c_i\alpha_i\right)=0$,于是$\sum c_i\alpha_i=0$,故由$(\star)$,$c_i=0$.

D的列向量线性无关,故$\det(D)\neq 0$.

②若$\det(D)\neq 0$,则D的列向量线性无关,则'$\sum c_i\begin{bmatrix}(\alpha_1,\alpha_i)\\(\alpha_2,\alpha_i)\\\vdots\\(\alpha_s,\alpha_i)\end{bmatrix}=0\Rightarrow c_i=0$'......$(*)$

令$\sum c_i\alpha_i=0$,则$\left(\alpha_j,\sum c_i\alpha_i\right)=(\alpha_j,0)=0$,于是$\sum c_i\begin{bmatrix}(\alpha_1,\alpha_i)\\(\alpha_2,\alpha_i)\\\vdots\\(\alpha_s,\alpha_i)\end{bmatrix}=\begin{bmatrix}\left(\alpha_1,\sum c_i\alpha_i\right)\\\left(\alpha_2,\sum c_i\alpha_i\right)\\\vdots\\\left(\alpha_s,\sum c_i\alpha_i\right)\end{bmatrix}=0$,故由$(*)$,$c_i=0$.

10.将复方阵U分解为实部和虚部$U=P+iQ$(其中P,Q为实n阶方阵).

证明U为酉矩阵的充要条件是：$P'Q$对称,且$PP'+Q'Q=E$.

Pf: ①若U为酉矩阵,则$(P+iQ)(P'-iQ')=UU^*=E$

即$(PP'+QQ')+i(QP'-PQ')=E$

比较实部和虚部得到:$PP'+QQ'=E,QP'=PQ'=(QP')'$,得证!

②若$P'Q$对称,且$PP'+Q'Q=E$,则$P'Q=(P'Q)'=Q'P$.

于是$UU^*=(P+iQ)(P'-iQ')=(PP'+QQ')+i(QP'-PQ')=E$.得证!

16.设A是n维酉空间V内的一个线性变换,$A^*=-A$.证明：A的非零特征值都是纯虚数.

Pf:设λ_i是A的所有特征值,由于$A^*=-A$,故$A+A^*=0$,那么$\lambda_i+\bar{\lambda}_i$是$A+A^*$的特征值,所以$\lambda_i+\bar{\lambda}_i=0$

于是λ_i要么是0，要么是纯虚数.