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6. 证：记 $A$ 为秩等于 $r$ 的对称矩阵. 则 $A$ 合同相似于对角矩阵. 即 $A = C^T\mathrm{diag}\{a_1,\dots,a_r,0\dots0\}C$.
取 $r$ 个秩为1的矩阵为 $C^T\mathrm{diag}\{a_1,0,\dots,0\}C$, $C^T\mathrm{diag}\{0,a_2,0\dots0\}C$, $\dots$, $C^T\mathrm{diag}\{0,\dots0,a_r,0,\dots,0\}C$.
则 $A = C^Ta_1E_{11}C + C^Ta_2E_{22}C + \dots + C^Ta_rE_{rr}C$. □

7. 记 $\alpha_i = (a_{i1}, a_{i2}, \dots, a_{in})^T$. $X = (x_1, x_2, \dots, x_n)^T$　$i = 1,2,\dots,s$

则 $f = (X^T\alpha_1, X^T\alpha_2, \dots, X^T\alpha_s)\cdot(X^T\alpha_1, X^T\alpha_2, \dots, X^T\alpha_s)^T$

$= X^T(\alpha_1,\alpha_2,\dots,\alpha_s)\cdot(\alpha_1^T,\alpha_2^T,\dots,\alpha_s^T)^T X$

$= X^TA^TAX \cancel{=(AX)^TAX}$

则 $f$ 的秩 $= r(A^TA)$

由于. $Ax=0 \Rightarrow A^TAx=0$, $A^TAx=0 \Rightarrow x^TA^TAx=0 \Rightarrow (Ax)^TAx=0 \Rightarrow Ax=0$

故. $r(A^TA) = r(A)$ □

(⇒) 8. 若 $f = x^TAx$ 可以分解成 $f=gh$，其中 $g=\alpha^Tx$. $h=\beta^Tx$. $\alpha.\beta.x$ 都是列向量.

则 $f = gh = \alpha^Tx\cdot\beta^Tx = x^T\alpha\beta^Tx$. ~~由于 $\alpha\beta^T$ 与 $\alpha^T\beta$ 有相同~~

故 $r(A) = r(\alpha\beta^T) = 1$.

(⇐) 若 $r(A)=1$ 则. $A$ 可以写成2个列向量 $\alpha,\beta$ 的形式. $A=\alpha\beta^T$. 故 $g=\alpha^Tx$. $h=\beta^Tx$. $f=gh$

若 $r(A)=2$ 且符号差为0. 故 $A = C^T\mathrm{diag}\{1,-1,0,\dots0\}C$, $C$ 可逆. 则

$f = x^TC^T\mathrm{diag}\{1,-1,0,\dots0\}Cx$. 记 $y=Cx$. 则 $f = y^T\mathrm{diag}\{1,-1,0,\dots0\}y = y_1^2-y_2^2$

$=(y_1-y_2)(y_1+y_2)$ 是2个一次齐次多项式之积. □

4. 设 $F(k) = (kX_1+(1-k)X_2)' A (kX_1+(1-k)X_2)$，这是一个关于 $k$ 的多项式，$k\in[0,1]$

$F(0)=X_2'AX_2<0$，$F(1)=X_1'AX_1>0$ $\Rightarrow \exists k_0\in(0,1)$ s.t. $F(k_0)=0$

取 $X_0=k_0X_1+(1-k_0)X_2$ 即可 □

5. $l_i=\alpha_i^T X$．其中 $\alpha_i=(a_{i1},a_{i2},\dots,a_{in})^T$，$X=(x_1,x_2,\dots,x_n)^T$

则 $f = l_1^2+\dots+l_p^2-l_{p+1}^2-\dots-l_{p+q}^2 = (l_1,l_2,\dots,l_{p+q})^T \operatorname{diag}\{\underbrace{1,\dots,1}_{p个},\underbrace{-1,\dots,-1}_{q个}\}(l_1,\dots,l_{p+q})$

$= (\alpha_1^TX,\alpha_2^TX,\dots,\alpha_{p+q}^TX)^T \operatorname{diag}\{\underbrace{1,1,\dots,1}_{p个},\underbrace{-1,\dots,-1}_{q个}\}(\alpha_1^TX,\alpha_2^TX,\dots,\alpha_{p+q}^TX)$

$= X^T(\alpha_1,\alpha_2,\dots,\alpha_{p+q}) \operatorname{diag}\{\underbrace{1,1,\dots,1}_{p个},\underbrace{-1,\dots,-1}_{q个}\}(\alpha_1^T,\alpha_2^T,\dots,\alpha_{p+q}^T)^TX$．

由规范型理论：$f$ 正惯性指数 $\le p$．负惯性指数 $\le q$ □

6. 不妨设 $f=X'AX$ 为规范型，否则进行基的可逆变换即可，记 $f=X'AX$ 的正惯性指数为 $p$，则 $\dim M=$ "A对角线上为0的个数" $=n-p$． □

1.(1). $f=99x_1^2-12x_1x_2+48x_1x_3+130x_2^2-60x_2x_3+71x_3^2$

$$= \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}^T\begin{pmatrix}99&-6&24\\-6&130&-30\\24&-30&71\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

由于 $|99|>0$，$\begin{vmatrix}99&-6\\-6&130\end{vmatrix}>0$，$\begin{vmatrix}99&-6&24\\-6&130&-30\\24&-30&71\end{vmatrix}=755874>0$

故 $f$ 正定（顺序主子式都为正） □

1.(3). $f=\sum_{i=1}^n x_i^2+\sum_{1\le i<j\le n}x_ix_j = \begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}^T\begin{pmatrix}1&\frac12&\cdots&\frac12\\\frac12&1&\ddots&\vdots\\\vdots&\ddots&\ddots&\frac12\\\frac12&\cdots&\frac12&1\end{pmatrix}\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$

考虑 $D_n = \begin{vmatrix}1&\frac12&\cdots&\frac12\\\frac12&1&\ddots&\vdots\\\vdots&\ddots&\ddots&\frac12\\\frac12&\cdots&\frac12&1\end{vmatrix}_{n\times n} = \begin{vmatrix}1&&&&\\&1&\frac12&\cdots&\frac12\\&\frac12&1&\ddots&\vdots\\&\vdots&\ddots&\ddots&\frac12\\&\frac12&\cdots&\frac12&1\end{vmatrix}_{(n+1)\times(n+1)} = \begin{vmatrix}1&0&0&\cdots&0\\\frac{n+1}{2}&1&\frac12&\cdots&\frac12\\\frac{n+1}{2}&\frac12&\ddots&\ddots&\vdots\\\vdots&\vdots&\ddots&\ddots&\frac12\\\frac{n+1}{2}&\frac12&\cdots&\frac12&1\end{vmatrix}$

$= \begin{vmatrix}\frac{1}{n+1}&-\frac{1}{n+1}&\cdots&-\frac{1}{n+1}\\\frac12&\frac12&&0\\\vdots&&\ddots&\\\frac12&0&&\frac12\end{vmatrix} = \begin{vmatrix}1&-\frac{1}{n+1}&\cdots&-\frac{1}{n+1}\\&\frac12&&\\&&\ddots&\\&&&\frac12\end{vmatrix} = \frac{1}{2^n}>0$．故 $f$ 所有顺序主子式 $>0$．

$\Rightarrow f$ 正定 □

2.(1) $f=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}^T\begin{pmatrix}1&t&-1\\t&1&2\\-1&2&5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$．$\Rightarrow |1|=1>0$．$\begin{vmatrix}1&t\\t&1\end{vmatrix}=1-t^2$．$\begin{vmatrix}1&t&-1\\t&1&2\\-1&2&5\end{vmatrix}=-5t^2-4t$．

$f$ 正定 $\Leftrightarrow 1-t^2>0 \wedge -5t^2-4t>0 \Leftrightarrow -1<t<1 \wedge 0<t<\frac45 \Leftrightarrow 0<t<\frac45$ □

4. 设 $A$ 的所有特征值为 $\lambda_1,\dots,\lambda_n$．取 ~~$t>\max\{\lambda_1,\dots,\lambda_n\}$~~ $t+\min\{\lambda_1,\dots,\lambda_n\}>0$ 即有 $tE+A$ 的所有特征值 $>0$．$\Rightarrow A$ 正定 □

7. ~~不妨设A对~~ 反证：若 $\forall n$ 维向量 $X$．都有 $X'AX\ge0$．又 $A$ 对称．则．$A$ 半正定

则 $A$ 的所有特征值 $\ge0$，则 $|A|\ge0$ 这与 $|A|<0$ 矛盾！ □

8. $A$．$B$ 正定，不妨设 $A=I_n$．因为条件和结论在合同变换 $A\to C^TAC=I_n$．$B\to C^TBC$．下保持不变．~~因为B正定~~．$B$ 正定 $\Leftrightarrow B$ 的特征值全 $>0 \Rightarrow B+I_n$ 特征值全 $>0$．$\Leftrightarrow A+B$ 正定 □

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9. $f = n\sum_{i=1}^{n} x_i^2 - \left[\sum_{i=1}^{n} x_i\right]^2 = X^T \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & n-1 \end{pmatrix} X$，记 $D_k = \begin{vmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & n-1 \end{vmatrix}_{k\times k} = \begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & n-1 & -1 & \cdots & -1 \\ \vdots & -1 & n-1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 1 & -1 & \cdots & -1 & n-1 \end{vmatrix}_{(k+1)\times(k+1)}$

$= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & n & & 0 \\ \vdots & & \ddots & \\ 1 & 0 & & n \end{vmatrix}_{(k+1)\times(k+1)} = \begin{vmatrix} \frac{n-k}{n} & 1 & \cdots & 1 \\ 0 & n & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & n \end{vmatrix}_{(k+1)\times(k+1)} \begin{cases} >0 & \forall\ k \le n-1 \\ =0 & \text{if } k=n \end{cases}$

$\Rightarrow$ $f$.顺序主子式全 $\geq 0$ $\Rightarrow$ $f$ 半正定 □

12.(1). ~~A = diag~~ $f = -x_1^2 + 2x_2^2 + 2x_3^2 + \cdots + 2x_n^2$. $g = 2x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2$. 则有 $f, g$ 非正定.
但 $f+g$ 正定. □

(2). 引理：$p(A+B) \leq p(A) + p(B)$. 证明：$\begin{pmatrix} I_n \\ I_n \end{pmatrix}^T \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I_n \\ I_n \end{pmatrix} = A+B \Rightarrow p(A+B) \leq p\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = p(A) + p(B)$

原题，则有 $p(A+B) \leq p(A) + p(B) < \frac{n}{2} + \frac{n}{2} = n \Rightarrow p(A+B) \leq n-1 \Rightarrow f+g = X'(A+B)X$ 非正定 □