1.(1) $f(x_1,x_2,x_3)=2x_1^2+3x_2^2+3x_3^2+2ax_2x_3=\boldsymbol{x}\begin{pmatrix}2&&\\&3&a\\&a&3\end{pmatrix}\boldsymbol{x}^T$

$=\boldsymbol{y}\begin{pmatrix}1&&\\&2&\\&&5\end{pmatrix}\boldsymbol{y}^T$,其中 $\boldsymbol{x}=(x_1,x_2,x_3),\boldsymbol{y}=(y_1,y_2,y_3)$.

于是 $\begin{pmatrix}2&&\\&3&a\\&a&3\end{pmatrix}$ 正交合同于 $\begin{pmatrix}1&&\\&2&\\&&5\end{pmatrix}$

由于是正交变换,故 $\det\begin{pmatrix}1&&\\&2&\\&&5\end{pmatrix}=\det\begin{pmatrix}2&&\\&3&a\\&a&3\end{pmatrix}\Rightarrow 10=2(9-a^2)\Rightarrow a=2(a>0)$

$$\begin{pmatrix}2&&&1&&\\&3&2&&1&\\&2&3&&&1\end{pmatrix}\to\begin{pmatrix}2&&&1&&\\&3&0&&1&\\&2&3-\frac{4}{3}&&&1\end{pmatrix}\to\begin{pmatrix}2&&&1&&\\&3&&&1&\\&&\frac{5}{3}&&-\frac{2}{3}&1\end{pmatrix}\to\begin{pmatrix}1&&&\frac{1}{\sqrt{2}}&&\\&2&&&\sqrt{\frac{2}{3}}&\\&&5&&-\frac{2}{3}&\sqrt{3}\end{pmatrix}$$

于是 $\begin{pmatrix}2&&\\&3&2\\&2&3\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}&&\\&\sqrt{\frac{2}{3}}&\\&-\frac{2}{3}&\sqrt{3}\end{pmatrix}\begin{pmatrix}1&&\\&2&\\&&5\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&&\\&\sqrt{\frac{2}{3}}&\\&-\frac{2}{3}&\sqrt{3}\end{pmatrix}^T$.

该变换为 $\boldsymbol{x}=\boldsymbol{y}\begin{pmatrix}\frac{1}{\sqrt{2}}&&\\&\sqrt{\frac{2}{3}}&\\&-\frac{2}{3}&\sqrt{3}\end{pmatrix}\Leftrightarrow\boldsymbol{y}=\boldsymbol{x}\begin{pmatrix}\frac{1}{\sqrt{2}}&&\\&\sqrt{\frac{2}{3}}&\\&-\frac{2}{3}&\sqrt{3}\end{pmatrix}^T$

$\Leftrightarrow(y_1,y_2,y_3)=(x_1,x_2,x_3)\begin{pmatrix}\frac{1}{\sqrt{2}}&&\\&\sqrt{\frac{2}{3}}&-\frac{2}{3}\\&&\sqrt{3}\end{pmatrix}=\left(\frac{x_1}{\sqrt{2}},\sqrt{\frac{2}{3}}x_2,-\frac{2}{3}x_2+\sqrt{3}x_3\right)$.□

1.(2) $f(\boldsymbol{x})$ 在单位圆 $\boldsymbol{x}\boldsymbol{x}^T=1$ 上的最大最小值?

$f(\boldsymbol{x})=\boldsymbol{x}\begin{pmatrix}2&&\\&3&2\\&2&3\end{pmatrix}\boldsymbol{x}^T=:\boldsymbol{x}A\boldsymbol{x}^T$

由瑞丽商:$\max\frac{\boldsymbol{x}A\boldsymbol{x}^T}{\boldsymbol{x}\boldsymbol{x}^T}=\lambda_{\max}=5,\min\frac{\boldsymbol{x}A\boldsymbol{x}^T}{\boldsymbol{x}\boldsymbol{x}^T}=\lambda_{\min}=1$.

故 $f(\boldsymbol{x})$ 在单位圆 $\boldsymbol{x}\boldsymbol{x}^T=1$ 上的最大值为5

$f(\boldsymbol{x})$ 在单位圆 $\boldsymbol{x}\boldsymbol{x}^T=1$ 上的最小值为1.□

2.$A\in M_n(F)$,若$A^2+5A+6I=0$,证明A可对角化.

$(A+3I)(A+2I)=0\Rightarrow A$的特征值只有$-2$和$-3$

断言$r(A+2I)+r(A+3I)=n$.

断言$r\left((A+2I)^k\right)+r\left((A+3I)^l\right)=n,\forall k,l\in\mathbb{Z}_{>0}$

于是$r\left((A+2I)^k\right),r\left((A+3I)^l\right)$为常值,$\forall k,l\in\mathbb{Z}_{>0}$.

于是A特征值-2的几何重数等于$r(A+2I)$,即特征值-2的代数重数

A特征值-3的几何重数等于$r(A+3I)$,即特征值-3的代数重数

故A的特征值的几何重数都等于代数重数,故A可对角化.□

断言的证明:$r(A+2I)+r(A+3I)=n$

证明:$(x+2)$和$(x+3)$互素,由于F是域,从而是PID,故存在多项式$u(x),v(x)\in F[x]$,

使得$u(x)(x+2)+v(x)(x+3)=1$

故$u(A)(A+2I)+v(A)(A+3I)=I$.

考虑初等变换$\begin{pmatrix}A+2I & \\ & A+3I\end{pmatrix}\to\begin{pmatrix}A+2I & u(A)(A+2I)\\ & A+3I\end{pmatrix}$

$\to\begin{pmatrix}A+2I & u(A)(A+2I)+v(A)(A+3I)\\ & A+3I\end{pmatrix}=\begin{pmatrix}A+2I & I\\ & A+3I\end{pmatrix}\to\begin{pmatrix}O & I\\ -(A+3I)(A+2I) & A+3I\end{pmatrix}$

$\to\begin{pmatrix}O & I\\ -(A+3I)(A+2I) & O\end{pmatrix}=\begin{pmatrix}O & I\\ O & O\end{pmatrix}\Rightarrow r(A+2I)+r(A+3I)=r\begin{pmatrix}A+2I & \\ & A+3I\end{pmatrix}=r\begin{pmatrix}O & I\\ O & O\end{pmatrix}=n.$□

断言的证明:$r\left((A+2I)^k\right)+r\left((A+3I)^l\right)=n,\forall k,l\in\mathbb{Z}_{>0}$

证明:$\forall k,l\in\mathbb{Z}_{>0},(x+2)^k$和$(x+3)^l$互素,故存在存在多项式$u_k(x),v_l(x)\in F[x]$,

使得$u_k(x)(x+2)^k+v_l(x)(x+3)^l=1$

故$u_k(A)(A+2I)^k+v_l(A)(A+3I)^l=I$.

仿照上述证明就有$r\left((A+2I)^k\right)+r\left((A+3I)^l\right)=n,\forall k,l\in\mathbb{Z}_{>0}$.□

$3.A\begin{pmatrix}2\\2\\1\end{pmatrix}=\begin{pmatrix}2\\2\\1\end{pmatrix},A\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow A\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix},A\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}$

设 $A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$,则 $\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}a_{13}\\a_{23}\\a_{33}\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}\Rightarrow a_{13}=0,a_{23}=0,a_{33}=1.$

$A\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}=\begin{pmatrix}a_{11}+a_{12}\\a_{21}+a_{22}\\a_{31}+a_{32}\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}\Rightarrow a_{11}+a_{12}=a_{21}+a_{22}=1,a_{31}+a_{32}=0$

还有 $a_{ij}=a_{ji},i\neq j.$

于是 $A=\begin{pmatrix}a_{11}&1-a_{11}&0\\1-a_{22}&a_{22}&0\\0&0&1\end{pmatrix}=\begin{pmatrix}a_{11}&1-a_{11}&0\\1-a_{11}&a_{11}&0\\0&0&1\end{pmatrix}$

结合$\det(A)=1\cdot1\cdot(-1)=-1$可知$\det\begin{pmatrix}a_{11}&1-a_{11}&0\\1-a_{11}&a_{11}&0\\0&0&1\end{pmatrix}=a_{11}^2-(1-a_{11})^2=-1\Rightarrow a_{11}=0$

$\Rightarrow A=\begin{pmatrix}&1&\\1&&\\&&1\end{pmatrix}.\square$

4.3阶实方阵 A 的每个元素都与它的代数余子式相等,证明:A 为正交矩阵.

$A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}=\begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{pmatrix}=(A^*)^T.$

其中 $A\cdot A^*=\det A\cdot I\Rightarrow A\cdot A^T=\det A\cdot I\Rightarrow\det(A\cdot A^T)=\det(\det A\cdot I)\Rightarrow(\det(A))^2=(\det(A))^3$

$\Rightarrow\det(A)=1\Rightarrow A\cdot A^T=I$,故 A 为正交矩阵.$\square$

$5.A=(a_{ij})_{1\le i,j\le n},B=(b_{ij})_{1\le i,j\le n}$

故 $tr(AB^t)=tr((a_{ij})_{1\le i,j\le n}(b_{ji})_{1\le i,j\le n})=\sum_{i=1}^n\sum_{j=1}^n a_{ij}b_{ji}$

$tr(AA^t)=\sum_{i=1}^n\sum_{j=1}^n a_{ij}^2,tr(BB^t)=\sum_{i=1}^n\sum_{j=1}^n b_{ij}^2.$

$\Rightarrow tr(AA^t)tr(BB^t)=\left(\sum_{i=1}^n\sum_{j=1}^n a_{ij}^2\right)\left(\sum_{i=1}^n\sum_{j=1}^n b_{ij}^2\right)=\left(\sum_{i=1}^n\sum_{j=1}^n a_{ij}^2\right)\left(\sum_{i=1}^n\sum_{j=1}^n b_{ji}^2\right)$

$\geq\left[\sum_{i=1}^n\left(\sqrt{\sum_{j=1}^n a_{ij}^2\sum_{j=1}^n b_{ji}^2}\right)\right]^2\geq\left[\sum_{i=1}^n\sum_{j=1}^n a_{ij}b_{ji}\right]^2=(tr(AB^t))^2$

6.A为n级矩阵，证明:$tr(A) \leq \sqrt{n}\left(tr(AA^T)\right)^{\frac{1}{2}}$

记$A=(a_{ij})_{1\leq i,j\leq n}$，于是$tr(A)=\sum_{i=1}^{n} a_{ii}, tr(AA^T)=\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2$.

只需证$\left(\sum_{i=1}^{n} a_{ii}\right)^2 \leq n\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2$.

$$\left(\sum_{i=1}^{n} a_{ii}\right)^2 \leq \left(\sum_{i=1}^{n} a_{ii}^2\right)\left(\sum_{i=1}^{n} 1\right) \leq n\left(\sum_{i=1}^{n} a_{ii}^2\right) \leq n\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2.\Box$$

7.($\Rightarrow$)若存在可逆$P\in M_n(F)$使得P^tAP和P^tBP同时为对角矩阵Λ_1,Λ_2.且Λ_1可逆

于是$P^{-1}A^{-1}BP=P^{-1}A^{-1}(P^t)^{-1}P^tBP=(\Lambda_1)^{-1}\Lambda_2$为对角矩阵.于是$A^{-1}B$在数域$F$可对角化.

($\Leftarrow$)存在可逆Q使得$Q^{-1}A^{-1}BQ=\Lambda \Rightarrow \Lambda=Q^{-1}A^{-1}BQ=(Q^tAQ)^{-1}(Q^tBQ)$

因此不妨一开始就令$A^{-1}B$为对角矩阵.

$B=A(A^{-1}B)=A\Lambda, B^T=(A\Lambda)^T=\Lambda^TA^T=\Lambda A$

$B=B^T \Rightarrow A\Lambda=\Lambda A \Rightarrow B=A^{-1}BA \Rightarrow AB=BA$

断言 A,B可同时正交相似对角化.$\Box$

引理:A,B为n阶对称矩阵,$AB=BA$.则A,B可同时正交相似对角化.

证明:若A是数量矩阵,则显然.若A不是数量矩阵,

$n=1$时,显然成立.假设命题对$\leq n-1$都成立,则

$AB=BA \Rightarrow A$的特征子空间都是B的不变子空间,$V=\bigoplus_{i=1}^{s} V_i$.

归纳假设$\Rightarrow B|_{V_i}$和$A|_{V_i}$可同时正交相似对角化,$\forall i$.

于是存在V_i的一组标准正交基使得线性变换$B|_{V_i}$和$A|_{V_i}$在这组基下都是对角阵.

将所有V_i的标准正交基拼起来构成V的一组标准正交基,就有A,B在这组基下都是对角阵.

即A,B可同时正交相似对角化.$\Box$

8.$A-B$半正定$\Rightarrow A-B$对称$\overset{B正定}{\Rightarrow} A$实对称.

于是存在可逆C,使得$C^TAC=\Lambda, C^TBC=I$.

不妨一开始就令$A=\Lambda=\mathrm{diag}\{\lambda_1,\cdots,\lambda_n\}, B=I$.

于是$\Lambda-I$半正定$\Rightarrow \lambda_i-1\geq 0, i=1,\cdots,n$.

于是$|A-\lambda B|=|\Lambda-\lambda I|=0$的根$\lambda_0$都满足$\lambda_0\geq 1$,且$|\Lambda|=\prod_{i=1}^{n}\lambda_i \geq 1=|I|$.$\Box$

9.设$A_1,A_2,B\in M_n(\mathbb{R}),A=\begin{pmatrix}A_1 & B\\ B^t & A_2\end{pmatrix}$.如果$A$是正定的,那么$|B|^2\leq|A_1|\,|A_2|$.

A正定$\Rightarrow A_1,A_2$也正定$\Rightarrow$存在可逆C_1,C_2,使得$C_1^tA_1C_1=I_n,C_2^tA_2C_2=I_n$

于是$\begin{pmatrix}C_1^t & \\ & C_2^t\end{pmatrix}\begin{pmatrix}A_1 & B\\ B^t & A_2\end{pmatrix}\begin{pmatrix}C_1 & \\ & C_2\end{pmatrix}=\begin{pmatrix}I_n & C_1^tBC_2\\ C_2^tB^tC_1 & I_n\end{pmatrix}=\begin{pmatrix}I_n & C_1^tBC_2\\ (C_1^tBC_2)^t & I_n\end{pmatrix}$正定

由于$(C_1^tBC_2)^t$与I_n可交换,故$\det\begin{pmatrix}I_n & C_1^tBC_2\\ (C_1^tBC_2)^t & I_n\end{pmatrix}=|I_n|\,|I_n|-|C_1^tBC_2|\,|C_1^tBC_2|\geq 0$

$1-|C_1|^2|C_2|^2|B|^2\geq 0$

$\Rightarrow |A_1|\,|A_2|-|B|^2=\dfrac{1}{|C_1|^2|C_2|^2}\left(|C_1^tA_1C_1|\,|C_2^tA_2C_2|-|C_1|^2|C_2|^2|B|^2\right)=\dfrac{1-|C_1|^2|C_2|^2|B|^2}{|C_1|^2|C_2|^2}\geq 0.\Box$

10.$A=(\alpha_1,\cdots,\alpha_n)$的列向量都是单位向量,$|\det(A)|=\pm 1$.

$$A^TA=(\alpha_1,\cdots,\alpha_n)^T(\alpha_1,\cdots,\alpha_n)=\begin{pmatrix}\alpha_1^T\\ \vdots\\ \alpha_n^T\end{pmatrix}(\alpha_1,\cdots,\alpha_n)=\begin{pmatrix}\alpha_1^T\alpha_1 & \alpha_1^T\alpha_2 & \cdots & \alpha_1^T\alpha_n\\ \alpha_2^T\alpha_1 & \alpha_2^T\alpha_2 & \cdots & \alpha_2^T\alpha_n\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_n^T\alpha_1 & \alpha_n^T\alpha_2 & \cdots & \alpha_n^T\alpha_n\end{pmatrix}$$

$=\begin{pmatrix}1 & \alpha_1^T\alpha_2 & \cdots & \alpha_1^T\alpha_n\\ \alpha_2^T\alpha_1 & 1 & \cdots & \alpha_2^T\alpha_n\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_n^T\alpha_1 & \alpha_n^T\alpha_2 & \cdots & 1\end{pmatrix}$.$A^TA$正定,因为$A$非异.

由$Hadamard$不等式可知,$\det(A^TA)\leq 1\cdot\cdots\cdot 1=1$,又因为$\det(A^TA)=|\det(A)|^2=1$.

等号成立,故A^TA为对角矩阵,故$A^TA=I_n\Rightarrow A$为正交矩阵.$\Box$