10.8 作业

解析几何：

18.(1)

$$\begin{cases} x+y+z+3=0 \\ 2x+3y-z+1=0 \end{cases} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ -1 \end{pmatrix}=\begin{pmatrix} -3 \\ 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 5 \end{pmatrix}=\begin{pmatrix} -8 \\ 5 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x=-8+4z \\ y=5 \end{cases} \Rightarrow \text{直线点法式：}\frac{x+8}{4}=\frac{y-5}{0}=\frac{z}{1}$$

1.(1)

$$\begin{cases}2x_1-x_2+x_3-x_4=1\\2x_1-x_2+0x_3+3x_4=2\\3x_1+0x_2+x_3+x_4=-3\\2x_1+2x_2-2x_3+5x_4=-6\end{cases}\Leftrightarrow\begin{pmatrix}2&-1&1&-1\\2&-1&0&3\\3&0&1&1\\2&2&-2&5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\2\\-3\\-6\end{pmatrix}$$

$$\Leftrightarrow\begin{pmatrix}2&-1&1&-1\\0&0&-1&4\\0&3&-1&5\\0&3&-3&6\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&0&0&0\\-1&1&0&0\\-3&0&2&0\\-1&0&0&1\end{pmatrix}\begin{pmatrix}2&-1&1&-1\\2&-1&0&3\\3&0&1&1\\2&2&-2&5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&0&0&0\\-1&1&0&0\\-3&0&2&0\\-1&0&0&1\end{pmatrix}\begin{pmatrix}1\\2\\-3\\-6\end{pmatrix}=\begin{pmatrix}1\\1\\-9\\-7\end{pmatrix}$$

$$\Leftrightarrow\begin{pmatrix}2&0&0&1\\0&0&1&-4\\0&0&2&-1\\0&3&-3&6\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&0&0&\frac{1}{3}\\0&-1&0&0\\0&0&1&-1\\0&0&0&1\end{pmatrix}\begin{pmatrix}2&-1&1&-1\\0&0&-1&4\\0&3&-1&5\\0&3&-3&6\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&0&0&\frac{1}{3}\\0&-1&0&0\\0&0&1&-1\\0&0&0&1\end{pmatrix}\begin{pmatrix}1\\1\\-9\\-7\end{pmatrix}=\begin{pmatrix}-\frac{4}{3}\\-1\\-2\\-7\end{pmatrix}$$

$$\Leftrightarrow\begin{pmatrix}1&0&0&\frac{1}{2}\\0&1&-1&2\\0&0&1&-4\\0&0&0&7\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&0&0&0\\0&0&0&\frac{1}{3}\\0&1&0&0\\0&-2&1&0\end{pmatrix}\begin{pmatrix}2&0&0&1\\0&0&1&-4\\0&0&2&-1\\0&3&-3&6\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&0&0&0\\0&0&0&\frac{1}{3}\\0&1&0&0\\0&-2&1&0\end{pmatrix}\begin{pmatrix}-\frac{4}{3}\\-1\\-2\\-7\end{pmatrix}=\begin{pmatrix}-\frac{2}{3}\\-\frac{7}{3}\\-1\\0\end{pmatrix}$$

$$\Leftrightarrow\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&0&0&-\frac{1}{14}\\0&1&1&\frac{2}{7}\\0&0&1&\frac{4}{7}\\0&0&0&\frac{1}{7}\end{pmatrix}\begin{pmatrix}1&0&0&\frac{1}{2}\\0&1&-1&2\\0&0&1&-4\\0&0&0&7\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&0&0&-\frac{1}{14}\\0&1&1&\frac{2}{7}\\0&0&1&\frac{4}{7}\\0&0&0&\frac{1}{7}\end{pmatrix}\begin{pmatrix}-\frac{2}{3}\\-\frac{7}{3}\\-1\\0\end{pmatrix}=\begin{pmatrix}-\frac{2}{3}\\-\frac{10}{3}\\-1\\0\end{pmatrix}$$

$$\Leftrightarrow\begin{cases}x_1=-\frac{2}{3}\\x_2=-\frac{10}{3}\\x_3=-1\\x_4=0\end{cases}$$

1.(3)

$$\begin{cases} x_1+2x_2+0x_3-3x_4+2x_5=1 \\ x_1-x_2-3x_3+x_4-3x_5=2 \\ 2x_1-3x_2+4x_3-5x_4+2x_5=7 \\ 9x_1-9x_2+6x_3-16x_4+2x_5=25 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 2 & 0 & -3 & 2 \\ 1 & -1 & -3 & 1 & -3 \\ 2 & -3 & 4 & -5 & 2 \\ 9 & -9 & 6 & -16 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ 7 \\ 25 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 0 & -3 & 2 \\ 0 & -3 & -3 & 4 & -5 \\ 0 & -7 & 4 & 1 & -2 \\ 0 & -27 & 6 & 11 & -16 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}=\begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ -2 & & 1 & \\ -9 & & & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 & -3 & 2 \\ 1 & -1 & -3 & 1 & -3 \\ 2 & -3 & 4 & -5 & 2 \\ 9 & -9 & 6 & -16 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}=\begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ -2 & & 1 & \\ -9 & & & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 7 \\ 25 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 5 \\ 16 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 0 & -3 & 2 \\ 0 & -3 & -3 & 4 & -5 \\ 0 & -7 & 4 & 1 & -2 \\ 0 & 0 & 33 & -25 & 29 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & -9 & & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 & -3 & 2 \\ 0 & -3 & -3 & 4 & -5 \\ 0 & -7 & 4 & 1 & -2 \\ 0 & -27 & 6 & 11 & -16 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & -9 & & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 5 \\ 16 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 5 \\ 7 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 0 & -3 & 2 \\ 0 & -3 & -3 & 4 & -5 \\ 0 & 0 & 33 & -25 & 29 \\ 0 & 0 & 33 & -25 & 29 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & -7 & 3 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 & -3 & 2 \\ 0 & -3 & -3 & 4 & -5 \\ 0 & -7 & 4 & 1 & -2 \\ 0 & 0 & 33 & -25 & 29 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & -7 & 3 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 5 \\ 7 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 8 \\ 7 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 0 & -3 & 2 \\ 0 & -3 & -3 & 4 & -5 \\ 0 & 0 & 33 & -25 & 29 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 & -3 & 2 \\ 0 & -3 & -3 & 4 & -5 \\ 0 & 0 & 33 & -25 & 29 \\ 0 & 0 & 33 & -25 & 29 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 8 \\ 7 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 8 \\ -1 \end{pmatrix}$$

$\Rightarrow 0=-1$，矛盾！

故原方程无解！

1.(5)

$$\begin{cases}3x_1+4x_2-5x_3+7x_4=0\\2x_1-3x_2+3x_3-2x_4=0\\4x_1+11x_2-13x_3+16x_4=0\\7x_1-2x_2+x_3+3x_4=0\end{cases}\Leftrightarrow\begin{pmatrix}3&4&-5&7\\2&-3&3&-2\\4&11&-13&16\\7&-2&1&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow\begin{pmatrix}1&7&-8&9\\2&-3&3&-2\\4&11&-13&16\\7&-2&1&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&-1&&\\&1&&\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}3&4&-5&7\\2&-3&3&-2\\4&11&-13&16\\7&-2&1&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&-1&&\\&1&&\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow\begin{pmatrix}1&7&-8&9\\0&-17&19&-20\\0&-17&19&-20\\0&-51&57&-60\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&&&\\-2&1&&\\-4&&1&\\-7&&&1\end{pmatrix}\begin{pmatrix}1&7&-8&9\\2&-3&3&-2\\4&11&-13&16\\7&-2&1&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&&&\\-2&1&&\\-4&&1&\\-7&&&1\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow\begin{pmatrix}1&7&-8&9\\0&-17&19&-20\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&&&\\&1&&\\&-1&1&\\&-3&&1\end{pmatrix}\begin{pmatrix}1&7&-8&9\\0&-17&19&-20\\0&-17&19&-20\\0&-51&57&-60\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&&&\\&1&&\\&-1&1&\\&-3&&1\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow\begin{pmatrix}1&7&-8&9\\0&1&-\frac{19}{17}&\frac{20}{17}\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&&&\\&-\frac{1}{17}&&\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}1&7&-8&9\\0&-17&19&-20\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&&&\\&-\frac{1}{17}&&\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow\begin{pmatrix}x_1-\frac{3}{17}x_3+\frac{13}{17}x_4\\x_2-\frac{19}{17}x_3+\frac{20}{17}x_4\\0\\0\end{pmatrix}=\begin{pmatrix}1&0&-\frac{3}{17}&\frac{13}{17}\\0&1&-\frac{19}{17}&\frac{20}{17}\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&-7&&\\&1&&\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}1&7&-8&9\\0&1&-\frac{19}{17}&\frac{20}{17}\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&-7&&\\&1&&\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow\begin{cases}x_1=\frac{3}{17}x_3-\frac{13}{17}x_4\\x_2=\frac{19}{17}x_3-\frac{20}{17}x_4\end{cases}$$,其中x_3,x_4为自由未知量.

3.(2)

$$\begin{cases} 0x_1 + x_2 - x_3 + x_4 = 0 \\ 0x_1 - 7x_2 + 3x_3 + x_4 = 0 \\ x_1 + 3x_2 + 0x_3 - 3x_4 = 0 \\ x_1 - 2x_2 + 3x_3 - 4x_4 = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 0 & 1 & -1 & 1 \\ 0 & -7 & 3 & 1 \\ 1 & 3 & 0 & -3 \\ 1 & -2 & 3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

方法一：该方程有非零解等价于系数矩阵为奇异矩阵.

$$\det\begin{pmatrix} 0 & 1 & -1 & 1 \\ 0 & -7 & 3 & 1 \\ 1 & 3 & 0 & -3 \\ 1 & -2 & 3 & -4 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & 1 \\ -7 & 3 & 1 \\ -2 & 3 & -4 \end{pmatrix} - \det\begin{pmatrix} 1 & -1 & 1 \\ -7 & 3 & 1 \\ 3 & 0 & -3 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & -1 & 1 \\ -7 & 3 & 1 \\ 5 & 0 & -5 \end{pmatrix} - \det\begin{pmatrix} 1 & -1 & 1 \\ -7 & 3 & 1 \\ 3 & 0 & -3 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & -1 & 1 \\ -4 & 0 & 4 \\ 5 & 0 & -5 \end{pmatrix} - \det\begin{pmatrix} 1 & -1 & 1 \\ -4 & 0 & 4 \\ 3 & 0 & -3 \end{pmatrix} = 0 - 0 = 0,$$

故该方程有非零解

方法二:原方程 $\Leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} & & 1 & \\ & & -1 & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 & 1 \\ 0 & -7 & 3 & 1 \\ 1 & 3 & 0 & -3 \\ 1 & -2 & 3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 & -3 \\ 0 & -5 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & -7 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & & -3 & \\ & 1 & 5 & \\ & & 1 & \\ & & 7 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 & -3 \\ 0 & -5 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & -7 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & -6 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -4 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -\frac{1}{2} & & \\ & & & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 & -6 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -4 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & -6 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & & -3 & \\ & 1 & 1 & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 & -6 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 - x_4 \\ x_3 - 2x_4 \\ 0 \end{pmatrix}$$

$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = x_4 \\ x_3 = 2x_4 \end{cases}$,其中x_4为自由未知量.

故该方程有非零解

$$\begin{cases} x_1 - x_2 + 0x_3 + 0x_4 = 0 \\ 0x_1 + x_2 - x_3 + 0x_4 = 0 \\ 0x_1 + 0x_2 + x_3 - x_4 = 0 \\ -x_1 + 0x_2 + 0x_3 + x_4 = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

方法一：该方程有非零解等价于系数矩阵为奇异矩阵.

$$\det\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} - \det\begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} - \det\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = 0$$

故该方程有非零解

$$\text{方法二:原方程} \Leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - x_4 \\ x_2 - x_4 \\ x_3 - x_4 \\ 0 \end{pmatrix}$$

$\Leftrightarrow x_1 = x_2 = x_3 = x_4$,其中$x_4$为自由未知量.

故该方程有非零解

4.

$$\begin{cases}2x+y+z=0\\ ax+0y-z=0\\ -x+0y+3z=0\end{cases}\Leftrightarrow\begin{pmatrix}2&1&1\\a&0&-1\\-1&0&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

方法一：该方程有非零解等价于系数矩阵为奇异矩阵.

$$\det\begin{pmatrix}2&1&1\\a&0&-1\\-1&0&3\end{pmatrix}=-\det\begin{pmatrix}a&-1\\-1&3\end{pmatrix}=-(3a-1)=0\Leftrightarrow a=\frac{1}{3}$$

$a=\dfrac{1}{3}$时，该方程有非零解.

$$\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}1&&\\&&1\\&3&1\end{pmatrix}\begin{pmatrix}2&1&1\\\frac{1}{3}&0&-1\\-1&0&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2&1&1\\-1&0&3\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\Leftrightarrow\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&&\\\frac{1}{2}&1&\\&&1\end{pmatrix}\begin{pmatrix}2&1&1\\-1&0&3\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}&\frac{7}{2}\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\Leftrightarrow\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}1&-1&\\&2&\\&&1\end{pmatrix}\begin{pmatrix}1&\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}&\frac{7}{2}\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&0&-3\\0&1&7\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x-3z\\y+7z\\z\end{pmatrix}$$

$$\Leftrightarrow\begin{cases}x=3z\\y=-7z\end{cases},\text{其中}z\text{为自由未知量.}$$

方法二:原方程 $\Leftrightarrow\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}1&&2\\&1&\\&&-1\end{pmatrix}\begin{pmatrix}2&1&1\\a&0&-1\\-1&0&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0&1&7\\a&0&-1\\1&0&-3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$

$$\Leftrightarrow\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}&&1\\1&&\\&1&-a\end{pmatrix}\begin{pmatrix}0&1&7\\a&0&-1\\1&0&-3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&0&-3\\0&1&7\\0&0&3a-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x-3z\\y+7z\\(3a-1)z\end{pmatrix}$$

$\Leftrightarrow\begin{cases}x=3z\\y=-7z\\(3a-1)z=0\end{cases}$，则该方程有非零解等价于$z\neq0,3a-1=0$即$a=\dfrac{1}{3}$.

此时解为$\begin{cases}x=3z\\y=-7z\end{cases}$,其中$z$为自由未知量.

6.原方程 $\Leftrightarrow \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ -1 & & & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ & & & & \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ -1 & & & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_1+a_2+a_3+a_4+a_5 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} x_1 \\ x_2-x_5 \\ x_3-x_5 \\ x_4-x_5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & 1 & & & -1 \\ & & 1 & & -1 \\ & & & 1 & -1 \\ & & & & \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \\ & 1 & 1 & 1 & \\ & & 1 & 1 & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ & & & & \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \\ & 1 & 1 & 1 & \\ & & 1 & 1 & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_1+a_2+a_3+a_4+a_5 \end{pmatrix} = \begin{pmatrix} a_1+a_2+a_3+a_4 \\ a_2+a_3+a_4 \\ a_3+a_4 \\ a_4 \\ a_1+a_2+a_3+a_4+a_5 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} x_1 = a_1+a_2+a_3+a_4 \\ x_2-x_5 = a_2+a_3+a_4 \\ x_3-x_5 = a_3+a_4 \\ x_4-x_5 = a_4 \\ 0 = a_1+a_2+a_3+a_4+a_5 \end{cases}$$

故该方程有解的充要条件是$a_1+a_2+a_3+a_4+a_5$=0.

此时，方程的解为$\begin{cases} x_1 = a_1+a_2+a_3+a_4 \\ x_2 = x_5+a_2+a_3+a_4 \\ x_3 = x_5+a_3+a_4 \\ x_4 = x_5+a_4 \end{cases}$,其中$x_5$为自由未知量.

10.9 作业

1.(1)

$$5\begin{bmatrix} 3 & -2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}-\begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 2 & 2 \end{bmatrix}-\frac{1}{2}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 15 & -10 & -5 \\ 10 & 0 & 5 \\ -5 & 5 & 10 \\ 0 & 5 & 0 \end{bmatrix}-\begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 2 & 2 \end{bmatrix}-\begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{31}{2} & -10 & -\frac{9}{2} \\ 9 & -1 & 3 \\ -\frac{11}{2} & \frac{11}{2} & \frac{21}{2} \\ -2 & 3 & -2 \end{bmatrix}$$

10.11 作业

1.

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ -1 & -1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 2 & 2 & 3 & -1 \end{pmatrix}$$

$$let\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ -1 & -1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 2 & 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \beta = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 8 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 2 & 3 & -2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} & & 1 & \\ & 1 & 1 & \\ & & -1 & 1 \\ 1 & & & \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ -1 & -1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 2 & 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} & & 1 & \\ & 1 & 1 & \\ & & -1 & 1 \\ 1 & & & \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 4 \\ 3 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 1 & & -1 & \\ & & & 1 \\ & & 1 & -1 \\ & 1 & & \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 2 & 3 & -2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 1 & & -1 & \\ & & & 1 \\ & & 1 & -1 \\ & 1 & & \end{pmatrix}\begin{pmatrix} 4 \\ 5 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \\ 5 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} c_1 - \frac{5}{3}c_5 \\ c_2 + \frac{4}{3}c_5 \\ c_3 \\ c_4 - \frac{2}{3}c_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{5}{3} \\ 0 & 1 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 1 & & & 1 \\ & 1 & & -2 \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 1 & & & 1 \\ & 1 & & -2 \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} c_1 - \frac{5}{3}c_5 = 1 \\ c_2 + \frac{4}{3}c_5 = 1 \\ c_3 = 1 \\ c_4 - \frac{2}{3}c_5 = 1 \end{cases} \Leftrightarrow \begin{cases} c_1 = \frac{5}{3}c_5 + 1 \\ c_2 = -\frac{4}{3}c_5 + 1 \\ c_3 = 1 \\ c_4 = \frac{2}{3}c_5 + 1 \end{cases}$$，其中c_5为自由未知量.

故$\beta = \left(\frac{5}{3}c + 1\right)a_1 + \left(-\frac{4}{3}c + 1\right)a_2 + a_3 + \left(\frac{2}{3}c + 1\right)a_4 + ca_5, \forall c \in \mathbb{R}.$

3.(1)

$$(a_1 \quad a_2 \quad a_3 \quad a_4)=\begin{pmatrix}1&1&1&1\\3&3&-2&4\\5&2&1&1\\-4&-2&-1&1\\0&1&-1&-1\end{pmatrix}$$

$$a_1,a_2,a_3,a_4\text{线性无关}\Leftrightarrow(a_1 \quad a_2 \quad a_3 \quad a_4)\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=0\ iff\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=0$$

$$(a_1 \quad a_2 \quad a_3 \quad a_4)\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=0\Leftrightarrow\begin{pmatrix}1&1&1&1\\3&3&-2&4\\5&2&1&1\\-4&-2&-1&1\\0&1&-1&-1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=0$$

$$\Leftrightarrow\begin{pmatrix}1&1&1&1\\0&0&-5&1\\0&-3&-4&-4\\0&2&3&5\\0&1&-1&-1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=\begin{pmatrix}1&&&&\\-3&1&&&\\-5&&1&&\\4&&&1&\\&&&&1\end{pmatrix}\begin{pmatrix}1&1&1&1\\3&3&-2&4\\5&2&1&1\\-4&-2&-1&1\\0&1&-1&-1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=0$$

$$\Leftrightarrow\begin{pmatrix}1&1&1&1\\0&1&-1&-1\\0&0&-7&-7\\0&0&5&7\\0&0&-5&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=\begin{pmatrix}1&&&&\\&&&&1\\&&1&&3\\&&&1&-2\\&1&&&\end{pmatrix}\begin{pmatrix}1&1&1&1\\0&0&-5&1\\0&-3&-4&-4\\0&2&3&5\\0&1&-1&-1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=0$$

$$\Leftrightarrow\begin{pmatrix}1&0&2&2\\0&1&-1&-1\\0&0&1&\frac{7}{5}\\0&0&0&8\\0&0&0&-42\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=\begin{pmatrix}1&-1&&&\\&1&&&\\&&&\frac{1}{5}&\\&&&1&1\\&&5&&-7\end{pmatrix}\begin{pmatrix}1&1&1&1\\0&1&-1&-1\\0&0&-7&-7\\0&0&5&7\\0&0&-5&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=0$$

$$\Leftrightarrow\begin{pmatrix}1&0&0&-\frac{4}{5}\\0&1&0&\frac{2}{5}\\0&0&1&\frac{7}{5}\\0&0&0&1\\0&0&0&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=\begin{pmatrix}1&&-2&&\\&1&1&&\\&&1&&\\&&&\frac{1}{8}&\\&&&&-\frac{1}{42}\end{pmatrix}\begin{pmatrix}1&0&2&2\\0&1&-1&-1\\0&0&1&\frac{7}{5}\\0&0&0&8\\0&0&0&-42\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=0$$

$$\Leftrightarrow\begin{pmatrix}c_1\\c_2\\c_3\\c_4\\0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=\begin{pmatrix}1&&&&\frac{4}{5}\\&1&&&-\frac{2}{5}\\&&1&&-\frac{7}{5}\\&&&&1\\&&&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&-\frac{4}{5}\\0&1&0&\frac{2}{5}\\0&0&1&\frac{7}{5}\\0&0&0&1\\0&0&0&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=0$$

$$\Leftrightarrow\begin{cases}c_1=0\\c_2=0\\c_3=0\\c_4=0\end{cases}\Leftrightarrow a_1,a_2,a_3,a_4\text{线性无关}.$$

5.

$$\because c_i \in \mathbb{R}, \begin{pmatrix} a_1 & a_2 & \cdots & a_s \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} a_1 & a_1 + a_2 & \cdots & a_1 + a_2 + \cdots + a_s \end{pmatrix} = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \cdots & a_s \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{pmatrix} = 0$$

$$\therefore a_1, a_2, \cdots, a_s \, are\ independent \Leftrightarrow \begin{pmatrix} a_1 & a_2 & \cdots & a_s \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{pmatrix} = 0 iff \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{pmatrix} = 0$$

$$\Leftrightarrow \begin{pmatrix} a_1 & a_1 + a_2 & \cdots & a_1 + a_2 + \cdots + a_s \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{pmatrix} = 0 iff \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{pmatrix} = 0 \Leftrightarrow a_1, a_1 + a_2, \cdots, a_1 + a_2 + \cdots + a_s \, are\ independent$$

$7.proof$:

记$a_1, a_2, \cdots, a_n$线性无关

$\forall m \in \{1, 2, \cdots, n-1\}$,下面证明$a_1, a_2, \cdots, a_n$中任意$m$项也线性无关，这$m$项记为$a_{n_1}, a_{n_2}, \cdots, a_{n_m}$.

构造矩阵$C = \left(c_{ij}\right)_{n \times m}$,其中$c_{ij} = \begin{cases} 1, 当 i = n_j 时 \\ 0, 当 i \neq n_j 时 \end{cases}$,

故$C^{-1} = \left(c'_{ji}\right)_{m \times n}$, 其中$c'_{ji} = \begin{cases} 1, 当 i = n_j 时 \\ 0, 当 i \neq n_j 时 \end{cases}$，$CC^{-1} = I$

$$a_1, a_2, \cdots, a_n 线性无关 \Leftrightarrow c_i \in \mathbb{R}, \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0 iff \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0$$

$$\Leftrightarrow \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} I \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0 iff \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} CC^{-1} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0 iff \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0$$

$$\Leftrightarrow \begin{pmatrix} a_{n_1} & a_{n_2} & \cdots & a_{n_m} \end{pmatrix} \begin{pmatrix} c_{n_1} \\ c_{n_2} \\ \vdots \\ c_{n_m} \end{pmatrix} = 0 iff \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} a_{n_1} & a_{n_2} & \cdots & a_{n_m} \end{pmatrix} \begin{pmatrix} c_{n_1} \\ c_{n_2} \\ \vdots \\ c_{n_m} \end{pmatrix} = 0 iff \begin{pmatrix} c_{n_1} \\ c_{n_2} \\ \vdots \\ c_{n_m} \end{pmatrix} = 0$$

$\Leftrightarrow a_{n_1}, a_{n_2}, \cdots, a_{n_m}$ 线性无关.

$9.proof:$

记$\{1,2,\cdots,n\}-\{i_1,i_2,\cdots,i_s\}=\{n_1,n_2,\cdots,n_{n-s}\}$,其中$n_1<n_2<\cdots<n_{n-s}$

$$(1)\{\alpha'_i\}_{1\le i\le m}\text{线性无关}\Leftrightarrow c_i\in\mathbb{R},\begin{cases}\sum_{i=1}^{m}c_ia_{in_1}=0\\ \sum_{i=1}^{m}c_ia_{in_2}=0\\ \quad\vdots\\ \sum_{i=1}^{m}c_ia_{in_{n-s}}=0\end{cases} iff\ c_i=0,\forall i\in\{1,2,\cdots,m\}$$

$$\sum_{i=1}^{m}c_i\alpha_i=0\Leftrightarrow\begin{cases}\sum_{i=1}^{m}c_ia_{i1}=0\\ \sum_{i=1}^{m}c_ia_{i2}=0\\ \quad\vdots\\ \sum_{i=1}^{m}c_ia_{in}=0\end{cases}\Leftrightarrow\begin{cases}\sum_{i=1}^{m}c_ia_{in_1}=0\\ \sum_{i=1}^{m}c_ia_{in_2}=0\\ \quad\vdots\\ \sum_{i=1}^{m}c_ia_{in_{n-s}}=0\end{cases}\wedge\begin{cases}\sum_{i=1}^{m}c_ia_{ii_1}=0\\ \sum_{i=1}^{m}c_ia_{ii_2}=0\\ \quad\vdots\\ \sum_{i=1}^{m}c_ia_{ii_s}=0\end{cases}\Leftrightarrow c_i=0,\forall i\in\{1,2,\cdots,m\}\wedge\begin{cases}\sum_{i=1}^{m}c_ia_{ii_1}=0\\ \sum_{i=1}^{m}c_ia_{ii_2}=0\\ \quad\vdots\\ \sum_{i=1}^{m}c_ia_{ii_s}=0\end{cases}\Rightarrow c_i=0,\forall i\in\{1,2,\cdots,m\}$$

$\Rightarrow\{\alpha_i\}_{1\le i\le m}$线性无关

$$(2)\{\alpha_i\}_{1\le i\le m}\text{线性相关}\Leftrightarrow\exists c_i\in\mathbb{R},c_i\text{不全为}0,\begin{cases}\sum_{i=1}^{m}c_ia_{i1}=0\\ \sum_{i=1}^{m}c_ia_{i2}=0\\ \quad\vdots\\ \sum_{i=1}^{m}c_ia_{in}=0\end{cases}\Rightarrow\exists c_i\in\mathbb{R},c_i\text{不全为}0,\begin{cases}\sum_{i=1}^{m}c_ia_{in_1}=0\\ \sum_{i=1}^{m}c_ia_{in_2}=0\\ \quad\vdots\\ \sum_{i=1}^{m}c_ia_{in_{n-s}}=0\end{cases}\Leftrightarrow\{\alpha'_i\}_{1\le i\le m}\text{线性相关}.$$