15．给定如下 4 阶方阵的行列式：

$$\begin{vmatrix} -1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ -3 & 1 & 0 & 2 \\ -1 & 1 & 0 & 1 \end{vmatrix},$$

求余子式 M_{13}, M_{31}, M_{24}.

$$\begin{vmatrix} -1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ -3 & 1 & 0 & 2 \\ -1 & 1 & 0 & 1 \end{vmatrix}$$

$$M_{13}=\begin{vmatrix} 0 & 1 & 1 \\ -3 & 1 & 2 \\ -1 & 1 & 1 \end{vmatrix}=\begin{vmatrix} 0 & 1 & 1 \\ -3 & 1 & 2 \\ -1 & 0 & 0 \end{vmatrix}=-1$$

$$M_{31}=\begin{vmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix}=\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix}=-1$$

$$M_{24}=\begin{vmatrix} -1 & 2 & 0 \\ -3 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix}=0$$

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(2) $\begin{vmatrix} 1 & -1 & 2 \\ 3 & 2 & 1 \\ 0 & 1 & 4 \end{vmatrix}$；

(4) $\begin{vmatrix} 1 & -2 & 1 & 0 \\ 0 & 3 & -2 & -1 \\ 4 & -1 & 0 & -3 \\ 1 & 2 & -6 & 3 \end{vmatrix}$；

$$(2)\begin{vmatrix}1 & -1 & 2\\3 & 2 & 1\\0 & 1 & 4\end{vmatrix}=-\begin{vmatrix}1 & 2\\3 & 1\end{vmatrix}+4\begin{vmatrix}1 & -1\\3 & 2\end{vmatrix}=5+20=25$$

$$(4)\begin{vmatrix}1 & -2 & 1 & 0\\0 & 3 & -2 & -1\\4 & -1 & 0 & -3\\1 & 2 & -6 & 3\end{vmatrix}=\begin{vmatrix}1 & -2 & 1 & 0\\0 & 3 & -2 & -1\\0 & 7 & -4 & -3\\0 & 4 & -7 & 3\end{vmatrix}=\begin{vmatrix}3 & -2 & -1\\7 & -4 & -3\\4 & -7 & 3\end{vmatrix}=\begin{vmatrix}3 & -2 & -1\\-2 & 2 & 0\\13 & -13 & 0\end{vmatrix}=-\begin{vmatrix}-2 & 2\\13 & -13\end{vmatrix}=0$$

$$(5)\begin{vmatrix}1 & 2 & -1 & 1\\3 & 0 & 1 & 2\\1 & -1 & 2 & 1\\1 & 0 & 3 & -2\end{vmatrix};$$

$$(5)\begin{vmatrix}1 & 2 & -1 & 1\\3 & 0 & 1 & 2\\1 & -1 & 2 & 1\\1 & 0 & 3 & -2\end{vmatrix}=\begin{vmatrix}3 & 0 & 3 & 3\\3 & 0 & 1 & 2\\1 & -1 & 2 & 1\\1 & 0 & 3 & -2\end{vmatrix}=\begin{vmatrix}3 & 3 & 3\\3 & 1 & 2\\1 & 3 & -2\end{vmatrix}=3\cdot\begin{vmatrix}1 & 1 & 1\\3 & 1 & 2\\1 & 3 & -2\end{vmatrix}=3\cdot\begin{vmatrix}1 & 1 & 1\\1 & -1 & 0\\3 & 5 & 0\end{vmatrix}=3\cdot\begin{vmatrix}1 & -1\\3 & 5\end{vmatrix}=24$$

22. 写出行列式$\begin{vmatrix}4 & -3 & 2 & 1 & -1\\0 & -5 & 3 & 0 & 1\\7 & -1 & 0 & 2 & -1\\1 & 0 & 1 & -1 & -1\\0 & 7 & 3 & 2 & 0\end{vmatrix}$对第三行的展开公式.

23. 证明

$$22.\begin{vmatrix}4 & -3 & 2 & 1 & -1\\0 & -5 & 3 & 0 & 1\\7 & -1 & 0 & 2 & -1\\1 & 0 & 1 & -1 & -1\\0 & 7 & 3 & 2 & 0\end{vmatrix}=7\begin{vmatrix}-3 & 2 & 1 & -1\\-5 & 3 & 0 & 1\\0 & 1 & -1 & -1\\7 & 3 & 2 & 0\end{vmatrix}+\begin{vmatrix}4 & 2 & 1 & -1\\0 & 3 & 0 & 1\\1 & 1 & -1 & -1\\0 & 3 & 2 & 0\end{vmatrix}+0-2\begin{vmatrix}4 & -3 & 2 & -1\\0 & -5 & 3 & 1\\1 & 0 & 1 & -1\\0 & 7 & 3 & 0\end{vmatrix}-\begin{vmatrix}4 & -3 & 2 & 1\\0 & -5 & 3 & 0\\1 & 0 & 1 & -1\\0 & 7 & 3 & 2\end{vmatrix}$$

19. 计算下列行列式：

(1) $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$； (2) $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$；

(3) $\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$；

(4) $\begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix}$；(5) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$；(6) $\begin{vmatrix} 0 & 1 & 1 & a \\ 1 & 0 & 1 & b \\ 1 & 1 & 0 & c \\ a & b & c & d \end{vmatrix}$.

19. (1) $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = 2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

$$= 2(x+y)\begin{vmatrix} 0 & 0 & 1 \\ y-x & y & x \\ x & x-y & y \end{vmatrix} = 2(x+y)\begin{vmatrix} y-x & y \\ x & x-y \end{vmatrix} = 2(x+y)\left[-(x-y)^2 - xy\right]$$

$$= -2(x+y)(x^2-xy+y^2) = -2x^3-2y^3$$

(2) $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} + \begin{vmatrix} x & 1 & 1 & 1 \\ & 1-x & 1 & 1 \\ & 1 & 1+y & 1 \\ & 1 & 1 & 1-y \end{vmatrix}$

$$= \begin{vmatrix} 1 & & & \\ 1 & -x & & \\ 1 & & y & \\ 1 & & & -y \end{vmatrix} + \begin{vmatrix} x & 1 & 1 & 1 \\ & 1-x & 1 & 1 \\ & 1 & 1+y & 1 \\ & 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} x & 1 & 1 & 1 \\ & 1-x & 1 & 1 \\ & 1 & 1+y & 1 \\ & 1 & 1 & y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & & & \\ 1 & -x & & \\ 1 & & y & \\ 1 & & & -y \end{vmatrix} + \begin{vmatrix} x & & & 1 \\ & -x & & 1 \\ & & y & 1 \\ & & & 1 \end{vmatrix} - \begin{vmatrix} x & 1 & 1 & \\ & 1-x & 1 & \\ & 1 & 1+y & \\ & 1 & 1 & y \end{vmatrix} = xy^2 - x^2y - xy\begin{vmatrix} 1-x & 1 \\ 1 & 1+y \end{vmatrix}$$

$$= xy^2 - x^2y - xy(-x+y-xy) = x^2y^2$$

(3) $\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = \begin{vmatrix} a^2 & 2a+1 & 4a+4 & 6a+9 \\ b^2 & 2b+1 & 4b+4 & 6b+9 \\ c^2 & 2c+1 & 4c+4 & 6c+9 \\ d^2 & 2d+1 & 4d+4 & 6d+9 \end{vmatrix} = \begin{vmatrix} a^2 & 2a+1 & 4a+4 & 6a+9 \\ b^2-a^2 & 2(b-a) & 4(b-a) & 6(b-a) \\ c^2-a^2 & 2(c-a) & 4(c-a) & 6(c-a) \\ d^2-a^2 & 2(d-a) & 4(d-a) & 6(d-a) \end{vmatrix}$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} a^2 & 2a+1 & 4a+4 & 6a+9 \\ a+b & 2 & 4 & 6 \\ a+c & 2 & 4 & 6 \\ a+d & 2 & 4 & 6 \end{vmatrix} = 6(b-a)(c-a)(d-a)\begin{vmatrix} a^2 & 2a+1 & 2a+2 & 2a+3 \\ a+b & 2 & 2 & 2 \\ a+c & 2 & 2 & 2 \\ a+d & 2 & 2 & 2 \end{vmatrix}$$

$$= 6(b-a)(c-a)(d-a)\begin{vmatrix} a^2 & 2a+1 & 1 & 2 \\ a+b & 2 & & \\ a+c & 2 & & \\ a+d & 2 & & \end{vmatrix} = 0$$

(4) $\begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix} = 6 \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix} = 6 \cdot \begin{vmatrix} 1 & & & \\ 1 & 2 & & \\ 1 & & 2 & \\ 1 & & & 2 \end{vmatrix} = 48$

26. 计算下列 n 阶行列式：

(1) $$\begin{vmatrix} a_1 & x & x & \cdots & \cdots & x \\ x & a_2 & x & \cdots & \cdots & x \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & x \\ x & \cdots & \cdots & \cdots & x & a_n \end{vmatrix};$$

(2) $$\begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & \cdots & n-1 & n & n \\ 3 & 4 & 5 & \cdots & n & n & n \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ n-1 & n & n & \cdots & n & n & n \\ n & n & n & \cdots & n & n & n \end{vmatrix};$$

(3) $$\begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 & \cdots & a_1b_n \\ a_1b_2 & a_2b_2 & a_2b_3 & \cdots & a_2b_n \\ a_1b_3 & a_2b_3 & a_3b_3 & \cdots & a_3b_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_1b_n & a_2b_n & a_3b_n & \cdots & a_nb_n \end{vmatrix};$$

(4) $$\begin{vmatrix} 7 & 5 & 0 & 0 & \cdots & \cdots & 0 \\ 2 & 7 & 5 & 0 & \cdots & \cdots & 0 \\ 0 & 2 & 7 & 5 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 5 \\ 0 & \cdots & \cdots & \cdots & 0 & 2 & 7 \end{vmatrix};$$

$$26.(1)\ \begin{vmatrix} a_1 & x & x & \cdots & \cdots & x \\ x & a_2 & x & \cdots & \cdots & x \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & x \\ x & \cdots & \cdots & \cdots & x & a_n \end{vmatrix} = \begin{vmatrix} a_1 & x-a_1 & x-a_1 & \cdots & \cdots & x-a_1 \\ x & a_2-x & & & & \\ \vdots & & \ddots & & & \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & \\ x & & & & & a_n-x \end{vmatrix}$$

$$= \begin{vmatrix} a_1+(x-a_1)\sum_{k=2}^{n}\frac{x}{x-a_k} & x-a_1 & x-a_1 & \cdots & \cdots & x-a_1 \\ & a_2-x & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & a_n-x \end{vmatrix} = \left(\prod_{k=2}^{n}(a_k-x)\right)\left(a_1+(x-a_1)\sum_{k=2}^{n}\frac{x}{x-a_k}\right)$$

$$= a_1\prod_{k=2}^{n}(a_k-x) + (x-a_1)\sum_{k=2}^{n}\frac{x}{x-a_k}\prod_{k=2}^{n}(a_k-x) = \prod_{k=2}^{n}(a_k-x)\left(a_1+(x-a_1)\sum_{k=2}^{n}\frac{x}{x-a_k}\right)$$

$$= -\prod_{k=1}^{n}(a_k-x)\left(\frac{a_1}{x-a_1}+\sum_{k=2}^{n}\frac{x}{x-a_k}\right) = -\prod_{k=1}^{n}(a_k-x)\left(\frac{x}{x-a_1}-1+\sum_{k=2}^{n}\frac{x}{x-a_k}\right)$$

$$= -\prod_{k=1}^{n}(a_k-x)\left(-1+\sum_{k=1}^{n}\frac{x}{x-a_k}\right)$$

$$(2)\ \begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & \cdots & n-1 & n & n \\ 3 & 4 & 5 & \cdots & n & n & n \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ n-1 & n & n & \cdots & n & n & n \\ n & n & n & \cdots & n & n & n \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 2 & 1 & \cdots & 1 & 1 & \\ 3 & 1 & & 1 & & \\ \vdots & \vdots & 1 & & & \\ n-1 & 1 & & & & \\ n & & & & & \end{vmatrix} = (-1)^{n}\cdot n$$

$$(3)\ \begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 & \cdots & a_1b_n \\ a_1b_2 & a_2b_2 & a_2b_3 & \cdots & a_2b_n \\ a_1b_3 & a_2b_3 & a_3b_3 & \cdots & a_3b_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_1b_n & a_2b_n & a_3b_n & \cdots & a_nb_n \end{vmatrix} := D_n$$

$$D_n = b_n\begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 & \cdots & a_1 \\ a_1b_2 & a_2b_2 & a_2b_3 & \cdots & a_2 \\ a_1b_3 & a_2b_3 & a_3b_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1b_n & a_2b_n & a_3b_n & \cdots & a_n \end{vmatrix} = b_n\begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 & \cdots & a_1 \\ a_1b_2 & a_2b_2 & a_2b_3 & \cdots & a_2 \\ a_1b_3 & a_2b_3 & a_3b_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & & \vdots \\ & & & \cdots & a_n \end{vmatrix} + b_n\begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 & \cdots & a_1 \\ a_1b_2 & a_2b_2 & a_2b_3 & \cdots & a_2 \\ a_1b_3 & a_2b_3 & a_3b_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1b_n & a_2b_n & a_3b_n & \cdots & 0 \end{vmatrix}$$

$$= a_nb_nD_{n-1} + b_n^2\begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 & \cdots & a_1 \\ a_1b_2 & a_2b_2 & a_2b_3 & \cdots & a_2 \\ a_1b_3 & a_2b_3 & a_3b_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & 0 \end{vmatrix}$$

Unsolved...

$$(4)\begin{vmatrix} 7 & 5 & & & & & \\ 2 & 7 & 5 & & & & \\ & 2 & 7 & 5 & & & \\ & & 2 & 7 & 5 & & \\ & & & 2 & \ddots & \ddots & \\ & & & & \ddots & \ddots & 5 \\ & & & & & 2 & 7 \end{vmatrix} := D_n$$

$$D_n = 7D_{n-1} - 2\cdot 5D_{n-2} = 7D_{n-1} - 10D_{n-2}$$

$$\Rightarrow \begin{cases} D_n - 2D_{n-1} = 5(D_{n-1} - 2D_{n-2}) \\ D_n - 5D_{n-1} = 2(D_{n-1} - 5D_{n-2}) \end{cases}$$

$$\Rightarrow \frac{D_n - 2D_{n-1}}{D_n - 5D_{n-1}} = \frac{5}{2}\frac{D_{n-1} - 2D_{n-2}}{D_{n-1} - 5D_{n-2}} = \cdots = \left(\frac{5}{2}\right)^{n-1}\frac{D_2 - 2D_1}{D_2 - 5D_1}$$

$$D_1 = 7, D_2 = \begin{vmatrix} 7 & 5 \\ 2 & 7 \end{vmatrix} = 39$$

$$\Rightarrow \frac{D_n - 2D_{n-1}}{D_n - 5D_{n-1}} = \left(\frac{5}{2}\right)^{n-1}\frac{D_2 - 2D_1}{D_2 - 5D_1} = \left(\frac{5}{2}\right)^{n-1}\frac{32}{4} = 8\cdot\left(\frac{5}{2}\right)^{n-1}$$

$$\Rightarrow D_n - 2D_{n-1} = 8\cdot\left(\frac{5}{2}\right)^{n-1}\cdot(D_n - 5D_{n-1}) \Rightarrow D_n = \frac{40\cdot\left(\frac{5}{2}\right)^{n-1} - 2}{8\cdot\left(\frac{5}{2}\right)^{n-1} - 1}D_{n-1}$$

$$\Rightarrow D_n = \prod_{k=2}^{n}\frac{40\cdot\left(\frac{5}{2}\right)^{n-1} - 2}{8\cdot\left(\frac{5}{2}\right)^{n-1} - 1}D_1 = 7\cdot\prod_{k=2}^{n}\frac{40\cdot\left(\frac{5}{2}\right)^{n-1} - 2}{8\cdot\left(\frac{5}{2}\right)^{n-1} - 1}$$

(5) $$\begin{vmatrix} 2\cos\alpha & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2\cos\alpha & 1 & \ddots & & \vdots \\ 0 & 1 & 2\cos\alpha & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 2\cos\alpha \end{vmatrix};$$

(6) $$\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & \cdots & a_n \\ -x_1 & x_2 & 0 & \cdots & \cdots & 0 \\ 0 & -x_2 & x_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -x_{n-1} & x_n \end{vmatrix}.$$

$$(5)\begin{vmatrix} 2\cos\alpha & 1 & & & & \\ 1 & 2\cos\alpha & 1 & & & \\ & 1 & 2\cos\alpha & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 2\cos\alpha \end{vmatrix} := D_n$$

$$D_n = 2\cos\alpha \cdot D_{n-1} - D_{n-2} \Rightarrow D_n - 2\cos\alpha \cdot D_{n-1} + D_{n-2} = 0$$

$$D_1 = 2\cos\alpha, D_2 = 4\cos^2\alpha - 1$$

考虑关于$\{D_n\}$的递推公式的特征方程:$x^2 - 2\cos\alpha \cdot x + 1 = 0$

$$\text{解得} x = \frac{2\cos\alpha \pm \sqrt{4 - 4\cos^2\alpha}}{2} = \cos\alpha \pm i\sin\alpha = e^{\pm i\alpha}$$

$$\Rightarrow D_n - (e^{i\alpha} + e^{-i\alpha}) \cdot D_{n-1} + (e^{i\alpha} \cdot e^{-i\alpha}) D_{n-2} = 0$$

$$D_1 = e^{i\alpha} + e^{-i\alpha}, D_2 = (e^{i\alpha} + e^{-i\alpha})^2 - 1 = e^{2i\alpha} + e^{-2i\alpha} + 1$$

$$D_n - e^{i\alpha} \cdot D_{n-1} = e^{-i\alpha}(D_{n-1} - e^{i\alpha} D_{n-2}) = \cdots = e^{-i(n-1)\alpha}(D_2 - e^{i\alpha} \cdot D_1)$$

$$= e^{-i(n-1)\alpha}(e^{2i\alpha} + e^{-2i\alpha} + 1 - e^{i\alpha} \cdot (e^{i\alpha} + e^{-i\alpha})) = e^{-i(n-1)\alpha} \cdot e^{-2i\alpha} = e^{-i(n+1)\alpha}$$

$$\Rightarrow D_n - e^{i\alpha} \cdot D_{n-1} = e^{-i(n+1)\alpha}$$

$$\Rightarrow D_n = e^{i\alpha} \cdot D_{n-1} + e^{-i(n+1)\alpha} = e^{i\alpha} \cdot (e^{i\alpha} \cdot D_{n-2} + e^{-in\alpha}) + e^{-i(n+1)\alpha}$$

$$= \cdots = e^{i(n-1)\alpha} \cdot D_1 + \sum_{k=0}^{n-2} e^{-i(n+1-2k)\alpha} = e^{i(n-1)\alpha} \cdot D_1 + e^{-i(n+1)\alpha} \sum_{k=0}^{n-2} e^{i2k\alpha}$$

$$= e^{i(n-1)\alpha} \cdot (e^{i\alpha} + e^{-i\alpha}) + e^{-i(n+1)\alpha} \frac{1 - e^{i2(n-1)\alpha}}{1 - e^{i2\alpha}}$$

$$= e^{in\alpha} + e^{i(n-2)\alpha} + \frac{e^{-i(n+1)\alpha} - e^{i(n-3)\alpha}}{1 - e^{i2\alpha}}$$

Unsolved...

$$(6)\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & \cdots & a_n \\ -x_1 & x_2 & 0 & \cdots & \cdots & 0 \\ 0 & -x_2 & x_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & -x_{n-1} & x_n \end{vmatrix} := D_n$$

$$D_n = x_n D_{n-1} + x_{n-1}\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_n \\ -x_1 & x_2 & 0 & \cdots & \cdots & 0 \\ 0 & -x_2 & x_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & x_{n-2} & 0 \\ 0 & \cdots & \cdots & 0 & -x_{n-2} & 0 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_n \\ -x_1 & x_2 & 0 & \cdots & \cdots & 0 \\ 0 & -x_2 & x_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & x_{n-2} & 0 \\ 0 & \cdots & \cdots & 0 & -x_{n-2} & 0 \end{vmatrix} = a_n\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ -x_1 & x_2 & 0 & \cdots & \cdots & 0 \\ 0 & -x_2 & x_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & x_{n-2} & 0 \\ 0 & \cdots & \cdots & 0 & -x_{n-2} & 0 \end{vmatrix} = (-1)^n a_n\begin{vmatrix} -x_1 & x_2 & & \\ & -x_2 & \ddots & \\ & & \ddots & x_{n-2} \\ & & & -x_{n-2} \end{vmatrix} = a_n\prod_{k=1}^{n-2}x_k$$

$$则 D_n = x_n D_{n-1} + x_{n-1}\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_n \\ -x_1 & x_2 & 0 & \cdots & \cdots & 0 \\ 0 & -x_2 & x_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & x_{n-2} & 0 \\ 0 & \cdots & \cdots & 0 & -x_{n-2} & 0 \end{vmatrix} = x_n D_{n-1} + a_n\prod_{k=1}^{n-1}x_k, D_1 = a_1$$

$$\Rightarrow \frac{D_n}{\prod_{k=1}^{n}x_k} = \frac{D_{n-1}}{\prod_{k=1}^{n-1}x_k} + \frac{a_n}{x_n} \Rightarrow \frac{D_n}{\prod_{k=1}^{n}x_k} = \frac{D_1}{x_1} + \sum_{k=2}^{n}\frac{a_k}{x_k} = \sum_{k=1}^{n}\frac{a_k}{x_k}$$

$$\Rightarrow D_n = \sum_{k=1}^{n}\frac{a_k}{x_k}\cdot\prod_{k=1}^{n}x_k$$

2. 给定矩阵

$$A=\begin{bmatrix}-1 & 2 & 0\\ -3 & 1 & 1\\ 0 & 2 & 0\end{bmatrix},$$

求它的伴随矩阵 A^*.

3. 利用伴随矩阵求下列矩阵的逆矩阵：

(1) $A=\begin{bmatrix}1 & 1 & -1\\ 2 & 1 & 0\\ 1 & -1 & 0\end{bmatrix}$；　(2) $A=\begin{bmatrix}2 & 2 & 3\\ 1 & -1 & 0\\ -1 & 2 & 1\end{bmatrix}$；

$$2.A=\begin{bmatrix}-1 & 2 & 0\\ -3 & 1 & 1\\ 0 & 2 & 0\end{bmatrix}\Rightarrow A^*=\begin{bmatrix}-2 & 0 & -6\\ 0 & 0 & 2\\ 2 & 1 & 5\end{bmatrix}$$

$$3.(1)A=\begin{bmatrix}1 & 1 & -1\\ 2 & 1 & 0\\ 1 & -1 & 0\end{bmatrix}\Rightarrow A^*=\begin{bmatrix}0 & 0 & -3\\ 1 & 1 & 2\\ 1 & -2 & -1\end{bmatrix},|A|=\begin{vmatrix}1 & 1 & -1\\ 2 & 1 & 0\\ 1 & -1 & 0\end{vmatrix}=\begin{vmatrix}1 & 1 & -1\\ 3 & 0 & 0\\ 1 & -1 & 0\end{vmatrix}=-3\cdot\begin{vmatrix}1 & -1\\ -1 & 0\end{vmatrix}=3$$

$$\Rightarrow A^{-1}=\frac{A^*}{|A|}=\frac{A^*}{3}=\begin{bmatrix}-\frac{2}{3} & 0 & -2\\ 0 & 0 & \frac{2}{3}\\ \frac{2}{3} & \frac{1}{3} & \frac{5}{3}\end{bmatrix}$$

5. 证明：对 n 阶方阵 $A(n\geqslant 2)$，有 $|A^*|=|A|^{n-1}$.

6. 设 A 是 n 阶方阵，$n\geqslant 2$. 证明：

$$r(A^*)=\begin{cases}n, & 当\ r(A)=n,\\ 1, & 当\ r(A)=n-1,\\ 0, & 当\ r(A)<n-1.\end{cases}$$

7. 设 A,B,T 均为 n 阶实数方阵，T 可逆. 证明：

(1) 若 $B=T^{-1}AT$，则 $|B|=|A|$；

(2) 若 $B=T'AT$，且 $|A|>0$，则 $|B|>0$.

$$5.proof:\ A\cdot A^*=\begin{bmatrix}|A| & & & \\ & |A| & & \\ & & \ddots & \\ & & & |A|\end{bmatrix}$$

$$\Rightarrow |A\cdot A^*|=\begin{vmatrix}|A| & & & \\ & |A| & & \\ & & \ddots & \\ & & & |A|\end{vmatrix}\Rightarrow |A|\cdot|A^*|=|A|^n\Rightarrow |A^*|=|A|^{n-1}$$

$$6.proof:$$

$$when\ r(A)=n,\ A\cdot A^*=|A|\cdot I\ \Rightarrow n=r(|A|\cdot I_n)\le r(\ A\cdot A^*)\le r(A^*)\le n\Rightarrow r(A^*)=n$$

$$when\ r(A)=n-1,\ without\ loss\ of\ generality,\ we\ assume\ that\ A=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & 1 & 0\\ 0 & \cdots & 0 & 0\end{bmatrix}$$

$$then\ A^*=\begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & 0 & 0\\ 0 & \cdots & 0 & 1\end{bmatrix},\ r(A^*)=1$$

$$when\ r(A)\le n-2,\ without\ loss\ of\ generality,\ we\ assume\ that\ A=\begin{bmatrix}I_{n-m} & \\ & 0_{m\times m}\end{bmatrix}(m\ge 2)$$

$$then\ A^*=\boldsymbol{O},\ r(A^*)=0.$$

$$7.(1)proof:|B|=|T^{-1}AT|=|T^{-1}|\,|A|\,|T|=|A|\,|T^{-1}|\,|T|=|A|\,|T^{-1}T|=|A|\,|I|=|A|$$

$$7.(2)proof:|B|=|T'AT|=|T'|\,|A|\,|T|=|A|\cdot|T|^2>0$$

10. 利用克莱姆法则解下列线性方程组：

$$(1)\begin{cases}2x_1-x_2+3x_3+2x_4=6,\\ 3x_1-3x_2+3x_3+2x_4=5,\\ 3x_1-x_2-x_3+2x_4=3,\\ 3x_1-x_2+3x_3-x_4=4;\end{cases}$$

$$10.(1)\begin{bmatrix}2&-1&3&2\\3&-3&3&2\\3&-1&-1&2\\3&-1&3&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}6\\5\\3\\4\end{bmatrix}$$

$$\Delta=\begin{vmatrix}2&-1&3&2\\3&-3&3&2\\3&-1&-1&2\\3&-1&3&-1\end{vmatrix}=-70$$

$$\Delta_1=\begin{vmatrix}6&-1&3&2\\5&-3&3&2\\3&-1&-1&2\\4&-1&3&-1\end{vmatrix}=-70$$

$$\Delta_2=\begin{vmatrix}2&6&3&2\\3&5&3&2\\3&3&-1&2\\3&4&3&-1\end{vmatrix}=-70$$

$$\Delta_3=\begin{vmatrix}2&-1&6&2\\3&-3&5&2\\3&-1&3&2\\3&-1&4&-1\end{vmatrix}=-70$$

$$\Delta_4=\begin{vmatrix}2&-1&3&6\\3&-3&3&5\\3&-1&-1&3\\3&-1&3&4\end{vmatrix}=-70$$

$$\Rightarrow (x_1,x_2,x_3,x_4)=\left(\frac{\Delta_1}{\Delta},\frac{\Delta_2}{\Delta},\frac{\Delta_3}{\Delta},\frac{\Delta_4}{\Delta}\right)=(1,1,1,1)$$