at_hw1

✍️内容

Elements of algebraic topology (Munkres, James R) (Z-Library).pdf

9. Let K be a complex in $\mathbf{R}^N$. Show that $|K|$ is a subspace of $\mathbf{R}^N$ if and only if each point x of $|K|$ lies in an open set of $\mathbf{R}^N$ that intersects only finitely many simplices of K. Generalize to $\mathbf{E}^J$.

Proof:

$(\Leftarrow)$: $|K|$ is a subset of $\mathbb{R}^N$. And in general, the topology of $|K|$ is finer than the subspace topology inherited from $\mathbb{R}^N$, which means some closed sets in the first topology may fail to be closed in the second.

But if K is finite, the topologies will be the same, because for any closed set $A \in |K|$ under the first topology, $A \cap \sigma$ is closed in σ for each $\sigma \in K$, thus is closed in $\mathbb{R}^N$. Hence, $A = \bigcup_{\sigma \in K} (A \cap \sigma)$, the union of closed set may not be closed, but the union is finite, so A is closed in $\mathbb{R}^N$.

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$(\Rightarrow)$: Prove by contradiction. Suppose that there exists one point x in $|K|$ such that any open set U in $\mathbb{R}^N$ containing x intersects infinitely many simplices of K. Let $U = B_1(x)$, the open ball in $\mathbb{R}^N$ centered at x with radius 1. Pick $x_1 \in \sigma_1 \cap B_1(x) \setminus \{x\}$, where σ_1 is a simplices that intersects $B_1(x)$. Then let $U = B_2(x)$, pick $x_2 \in \sigma_2 \cap B_{1/2}(x) \setminus \{x\}$, where σ_2 is a simplices different from σ_1 that intersects $B_{1/2}(x)$... For $U = B_{1/n}(x)$, there always exists a simplices σ_n different from $\sigma_1, \ldots, \sigma_{n-1}$ since U intersects with infinitely many simplices of K, and let $x_n \in \sigma_n \cap B_{1/n}(x) \setminus \{x\}$. Therefore we have a sequence $x_1, x_2, \ldots, x_n, \ldots$. The set $\{x_n : n \in \mathbb{N}_{\geq 1}\}$ is closed under the topology of $|K|$, but not closed under the subspace topology inherited from $\mathbb{R}^N$ since the limit point x is not contained in the set.

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10. Show that the collection of all simplices in **R** of the form $[1/(n+1), 1/n]$ for n a positive integer, along with their vertices, is a complex whose polytope is the subspace $(0,1]$ of **R**.

Proof:

It's obvious that the collection of all simplices in $\mathbb{R}$ of the form $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ for $n \in \mathbb{N}_{\geq 1}$ is a simplicial complex, denoted by K, and $|K| = (0,1]$.

Apply the conclusion in exercise 9 to this question. We know that $\forall x \in |K|$, x lies in $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ for some $n \in \mathbb{N}_{\geq 1}$, there exists an open set $\left(\frac{1}{n+2}, \frac{1}{n-\frac{1}{2}}\right) \subset \mathbb{R}$, which intersects only the sets $\left[\frac{1}{n+2}, \frac{1}{n+1}\right], \left[\frac{1}{n+1}, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{1}{n-1}\right]$ (if these sets $\in K$). Hence $|K| = (0,1]$ is a subspace of $\mathbb{R}^N$.

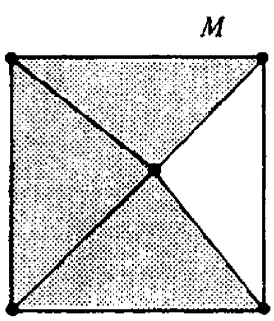


Figure 5.7

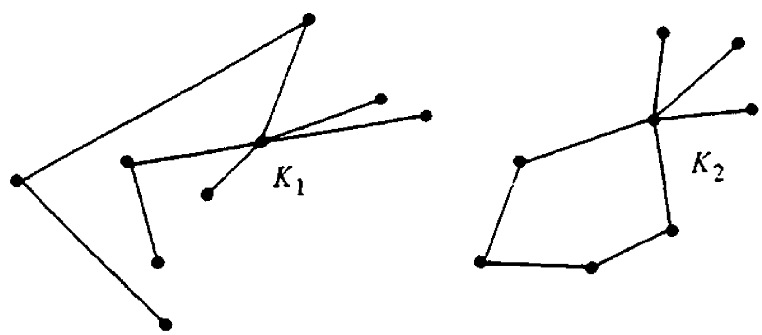


Figure 5.8

2. Consider the complex M pictured in Figure 5.7; it is the union of three triangles and a line segment. Compute the homology groups $H_1(M)$ and $H_2(M)$.

3. A 1-dimensional complex is called a **tree** if its 1-dimensional homology vanishes. Is either of the complexes pictured in Figure 5.8 a tree?

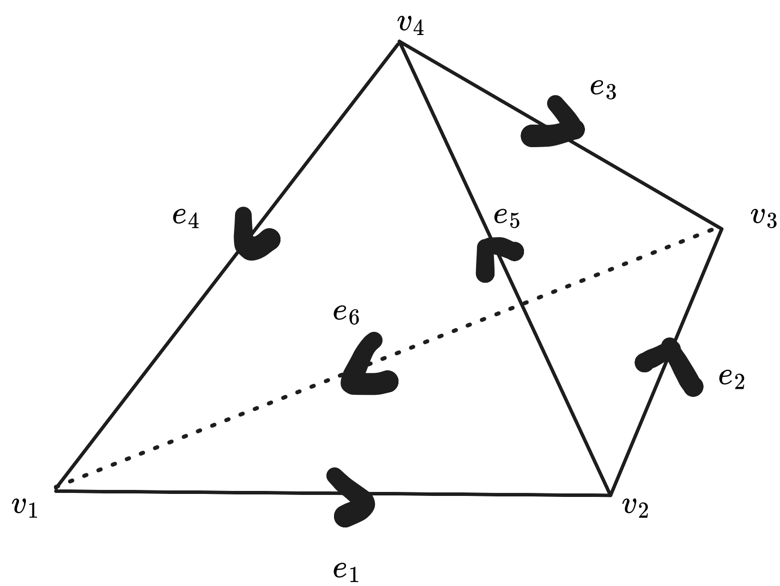
4. Let K be the complex consisting of the proper faces of a 3-simplex. Compute $H_1(K)$ and $H_2(K)$.

2 We know that $M \simeq S^1$, so $H_1(M) \simeq H_1(S^1) = \mathbb{Z}, H_2(M) \simeq H(S^1) = 0.$

3 We know that $K_1 \simeq \{x\}, K_2 \simeq S^1$, so $H_1(K_1) \simeq 0, H_1(K_2) \simeq \mathbb{Z}$. Hence K_1 is a tree, while K_2 is not.

4 We know that $K \simeq S^2$, so $H_1(K) \simeq 0, H_2(K) \simeq \mathbb{Z}$.

> **Note**
> 这很 tricky，其实我本来算了的，但是出了点小问题，交作业时间紧迫没有发现，先放下我算的。



算出
$Z_1(K) = \mathbb{Z}(e_1 + e_5 + e_4) + \mathbb{Z}(e_2 - e_3 - e_5) + \mathbb{Z}(e_3 + e_6 - e_4) + \mathbb{Z}(e_1 + e_2 + e_6), B_1(K$
, $\Rightarrow H_1(K) = Z_1(K)/B_1(K) = 0.$
$Z_2(K) = \mathbb{Z}([v_1, v_2, v_3], [v_3, v_2, v_4], [v_2, v_1, v_4], [v_1, v_3, v_4])$. 似乎 $Z_2(K) = B_2(K)$ 导致 $H_2(K) = 0$, 应该是我算错了.